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FDL-TDR-64-1
PART I, VOLUME 2

SIX-DEGREE-OF-FREEDOM FLIGHT PATH STUDY GENERALIZED COMPUTER PROGRAM

PART I, VOLUME 2 – STRUCTURAL LOADS FORMULATION

TECHNICAL DOCUMENTARY REPORT No. FDL-TDR-64-1
PART I, VOLUME 2

AUGUST 1964

AF FLIGHT DYNAMICS LABORATORY
RESEARCH AND TECHNOLOGY DIVISION
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE OHIO

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ABSTRACT

The equations of motion applicable to the Six-Degree-of-Freedom Structural Loads Program (SLP) are derived in this report. These equations are written for the determination of vehicle structural loads and response due to aerodynamic loads, loads due to control surface deflections, and environmental disturbances. Arbitrary elastic degrees of freedom (wing bending, wing torsion, body bending, etc.) and fuel slosh equations are incorporated into the overall analysis.

Newtonian flow theory is used for obtaining idealized aerodynamic pressure distributions since it is the simplest aerodynamic theory that offers sufficient generality. Accelerations, deflections, shear forces and bending moments at arbitrary stations can be computed.

This technical documentary report has been reviewed and is approved.

HOLLAND B. LOWNDES, JR.
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SYMBOLS

A^{rs}_j	static aerodynamic terms. See Equation (195)
$a_{ki\ell}$	given mode of vibration in degree of freedom k.
$a^r_{ki\ell}$	components of $\bar{e}_{i\ell}$ in the y coordinate system
a^{rs}_{ji}	inertia terms. See Equation (180).
B^{rs}_{jk}	aerodynamic stiffness terms. See Equation (196).
$\bar{b}_j$	the dynamically balancing rotation rate with respect to q^j of the vehicle relative to the vehicle axes.
b^r_j	components of $\bar{b}_j$ in the y coordinate system.
C^r_{jk}	aerodynamic damping terms. See Equation (197).
$\bar{c}_j$	the dynamically balancing translation rate with respect to q^j of the vehicle relative to the vehicle axes.
c^r_j	components of $\bar{c}_j$ in the y coordinate system.
c_{rst}	permutation symbol. See text preceding Equation (139).
E	number of thrust vectoring nozzles (or "engines").
e^r_{si}	components in the $\bar{J}_r$ system of the $\bar{J}'_{si}$ vectors.
$\bar{F}$	the sum of the external forces exerted on the vehicle.
$\bar{F}_{ih}$	the external force on the h-th particle of the i-th section.
$\bar{F}_{ihkj}$	the internal force exerted on the h-th particle of the i-th section by the j-th particle of the k-th section.
F_{ihkj}	the magnitude of $\bar{F}_{ihkj}$.

$\overline{G}$	the sum of the moments about the origin of the vehicle axes due to the external forces.
$\overline{g}$	the force per unit mass due to gravity.
γ'_j	the coefficient of "structural" damping associated with the j-th degree of freedom.
G_{rs}	products of inertia of structure and fuel about vehicle axes.
H'_{rsi}	the moments and the negatives of the products of inertia of section i about its own axes. See Equation (138).
H_{jk}	inertia coupling terms. See Equation (173).
H_k^r	modal unbalances. See Equation (150).
h	subscript used to denote a particle of a section.
$\overline{h}_{ji}$	the partial linear velocity with respect to q^j of the center of mass of section i relative to the vehicle axes - values obtained after dynamic balancing.
h^r_{ji}	components of $\overline{h}_{ji}$ in the y coordinate system.
I_{rs}	moments and negatives of products of inertia of structure and fuel about vehicle axes. See Equation (145).
i	subscript used to denote a section of the vehicle.
$\overline{J}_r$	three unit vectors pointing respectively in the directions of the three vehicle axes y^r. See Sec. 2 and Fig. 2.
$\overline{J}'_{ri}$	three unit vectors pointing respectively in the directions of the three axes η_i^r of section i. See Sec. 2 and 3 and Fig. 2.
$\overline{\jmath}_{ki}$	the partial linear velocity with respect to q^k of the center of mass of section i relative to the vehicle axes - arbitrary values given prior to dynamic balancing.
$\jmath^r_{ki}$	components of $\overline{\jmath}_{ki}$ in the y coordinate system.
j	suffix used to denote a degree of freedom.
K_{jk}	components of the stiffness tensor. See Equation (90).

h	suffix used to denote a degree of freedom
L_j^{r}	modal inertia terms. See Equation (15)
l	suffix used to denote a degree of freedom.
M_{JK}	components of the inertia tensor. See Equation (32)
$\bar{M}$	the bending moment at a specified location.
M_r	components of $\bar{M}$ in the y coordinate system.
m	total mass of vehicle and fuel at any instant.
m_i	mass of section i.
m_{ih}	mass of the h-th particle of section i.
N	the number of sections and tanks.
N_j	the generalized forces associated with inertia forces. See Equation (46).
N_j^r	modal inertia terms. See Equation (172).
n	number of elastic degrees of freedom.
$\bar{n}$	a unit vector located at a certain point on the surface, perpendicular to the surface at that point, and pointing outward.
n^r	components of $\bar{n}$ in the ν' coordinate system.
O_j	the generalized forces associated with conservative internal forces.
$\bar{o}_i$	position vector locating the origin of the ν_i' coordinate system with respect to the y coordinate system. See Fig. 2.
o^r	components of $\bar{o}_i$ in the y coordinate system and coordinates of the center of mass of section i.
P_i	the number of particles in the i-th section or tank of fuel.

P_j	the generalized forces associated with dissipative internal forces.
p_{rs}	modal moments and negatives of products of inertia of the vehicle. See Equation (53).
$\mathcal{P}_{ji}$	modal moments and negatives of products of inertia of section i. See Equation (179).
Q_j	the generalized forces associated with external forces. See Equation (47).
q^j	generalized coordinate associated with the j-th degree of freedom
r	suffix denoting the r-th coordinate axis in either the y or the ν_i system.
S_i	the surface of the i-th section.
s	suffix denoting the s-th coordinate axis in either the y or the ν_i system.
T	the kinetic energy of the vehicle and fuel.
T_i	the magntidue of the thrust force at the i-th nozzle.
t	time. Also used sometimes as a suffix in the same sense as r or s.
U	potential energy due to elastic deformation.
V	energy dissipated thru damping.
$\bar{V}$	linear velocity of the vehicle at the origin of the vehicle axes.
V^r	components of $\bar{V}$ in the y coordinate system.
$\bar{v}_{ih}$	velocity of the h-th particle of the i-th section.
W	the work done by the external forces.
w	the "piston speed" (or downwash) at a point on the surface.
$\bar{X}$	position vector of the vehicle in relation to a space-fixed frame of reference.
$\bar{y}_{ih}$	position vector of the h-th particle of the i-th section in relation to the vehicle axes.

y^r_{ih} — components of $\bar{y}_{ih}$ = y coordinates of the h-th particle of the ith section.

$\bar{y}_c$ — position vector of the center of mass of the vehicle.

y^r_c — components of $\bar{y}_c$ in the y coordinate system.

$\bar{\alpha}_{ji}$ — the partial angular velocity with respect to q^j of the η'_i coordinates relative to the y system - values obtained after dynamic balancing.

α'^r_{ji} — components of $\bar{\alpha}_{ji}$ in the η'_i coordinate system.

$\bar{\beta}_{ji}$ — the partial angular velocity with respect to q^j of the η'_i coordinates relative to the y system - arbitrary values given prior to dynamic balancing.

β'^r_{ji} — components of $\bar{\beta}_{ji}$ in the η'_i coordinate system.

Γ'_{ri} — products of inertia of section i referred to the sectional axes.

Δ_{ihkj} — the distance from particle kj to particle ih. See Equation (77).

Δ_{jk} — inertia coupling terms. See Equation (171).

δ_{rs} — the Kronecker delta

$\delta_{rs} = 1$ when $r = s$.

$\delta_{rs} = 0$ when $r \neq s$.

δ_j — the logarithmic decrement associated with the j-th degree of freedom.

δ_i	the angle of rotation of $\bar{j}'_{21}$ and $\bar{j}'_{31}$ about $j_{..}$. See Sec. 9.
$\zeta_{...}^{rs}$	inertia coupling terms. See Equation (173).
η_{jk}	inertia coupling terms. See Equation (169).
Λ_{ji}^{rs}	modal products of inertia of section i. See Equation (142).
λ_i	angle of swivel of nozzle (or the $\bar{j}_{1i}$ vector) about an axis ($\bar{L}_3$) perpendicular to $\bar{j}_1$ and making an angle ϕ_i with $\bar{j}_3$. See Sec. 9.
μ_{jk}	inertia coupling terms. See Equation (167).
ξ_j	aerodynamic modal term. See Equation (191).
π	ratio of circumference to diameter of a circle.
ρ	the atmospheric density.
$\bar{\sigma}_{jih}$	the partial linear velocity with respect to q^j of particle h relative to section i.
σ_{jih}^r	components of $\bar{\sigma}_{jih}$ in the ν_i^r coordinate system.
$\bar{\nu}_{ih}$	position vector of the h-th particle of the i-th section relative to the origin of the ν_i^r coordinate system.
ν_{ih}^r	components of $\bar{\nu}_{ih}$ in the ν_i^r system.
ϕ_i	angle of rotation of the axis and plane of swivel about the y^1 axis ($\bar{j}_1$). See Sec. 9.
$\bar{\Omega}$	angular velocity of the vehicle axes.
Ω^r	components of $\bar{\Omega}$ in the y coordinate system.
ω_j	vibration frequency associated with the j-th degree of freedom. See Equation (91).
$\boxed{KL,j}$	inertia "symbols". See Equation (59).

1. INTRODUCTION

This report includes the derivation of the equations to be used in the Structural Loads Program (SLP). This program is to be used in conjunction with the basic Six-Degree-of-Freedom Flight Path Computer Program (SDF), as a means to determine the vehicle structural loads and response due to aerodynamic loads, loads due to control surface deflections, and environmental disturbances (i.e., wind profiles and continuous discrete turbulence profiles). The program permits the inclusion of up to 17 elastic degrees of freedom and 40 fuel slosh modes. The elastic degrees of freedom are arbitrary, and the user may incorporate any number of modes such as body bending, wing bending, wing torsion, etc., so as to total 17. The fuel slosh modes incorporate 2 longitudinal and 2 lateral modes on each tank and the program allows one to include up to 10 tanks. It is recognized that the gross vehicle motion (large motions) influences the small motions (elastic deformations and fuel sloshing) of the vehicle, but it is assumed that these smaller motions have a negligible effect on the larger motions of the vehicle. Other basic assumptions used in this analysis are:

1. Undamped free vibration modes are used to specify the elastic deformations and fuel slosh.

2. There is no elastic or damping coupling between the degrees of freedom.

3. The aerodynamic forces can be obtained by Newtonian flow theory.

4. The fuel surface (except for the sloshing) is considered to be perpendicular to the resultant acceleration at the center of the tank.

5. The fuel slosh modes of a tank that is not vertical or horizontal can be represented by those of some hypothetical tank that is vertical or horizontal. In addition, longitudinal fuel sloshing in a horizontal cylindrical tank is represented by an analogy to a rectangular tank.

6. The effect of a rocket engine can be represented by a thrust vector, which is a simplification that assumes the center of mass flow through the nozzle to be exactly alined with the geometric axis of the nozzle.

The complexities inherent in this type of problem are so great that certain conventions of the tensor notation are incorporated in the subsequent development in order to shorten the writing of the equations. These operations are explicitly explained as they are introduced. The analysis of the structural loads is logically developed in the following sequence:

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1. The vehicle kinematics are derived.

2. The force and moment relations are found using Newtonian mechanics.

3. The equations of motion of the elastic deformation are derived (work and energy concepts are used to check the basic formulations of Items 2 and 3).

4. The main equations to be used to determine the elastic deformations are put into terms suitable for computation.

5. The aerodynamic forces (using Newtonian flow theory) are found.

6. The analysis of the fuel slosh problem is included.

7. The thrust forces are introduced.

8. The accelerations at all locations are found.

9. The shear forces and bending moments are calculated.

The generalized forces to be used in the program are inertia forces N_j, external forces Q_j, conservative internal forces O_j and dissipative internal forces P_j. These forces are represented by Equations (46) - (49).

To clarify to some degree the subsequent analysis, the representation of the coordinate system is presented in Figure 1. The origin of the orthogonal reference frame (y^1, y^2, y^3) is represented by an arbitrary point that would be fixed in the vehicle if it remained rigid during the motion along its flight path. In conjunction with this frame of reference, are located relative coordinate systems ($\upsilon'^1, \upsilon'^2, \upsilon'^3$) positioned at various points on the body to define the elastic deformations and fuel slosh motions. An absolute reference frame (X^1, X^2, X^3) is shown for generality with $\bar{X}$ the position vector connecting the origins of the reference frames. As a physical insight into the relative relations of these coordinate systems, consider the case when the vehicle center of gravity (C.G.) is the origin of the y^1, y^2, y^3 triad; then, the velocity of this point is represented by $\bar{V}_{CG} = \frac{d}{dt}\bar{X}_{CG}$.

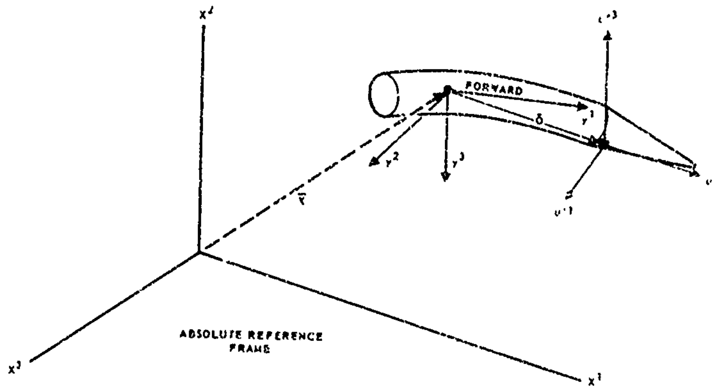


Figure 1. Coordinate Systems

2. KINEMATICS OF THE VEHICLE AND BASIC ASSUMPTIONS

In analyzing the motion of a vehicle in flight, it is convenient (perhaps necessary) to think in terms of the following types of motion: (1) the motion of the vehicle as a whole, which characterizes its "flight" and is referred to here as the gross motion of the vehicle, (2) large motions of certain parts, such as control surfaces, relative to the rest of the vehicle and large displacements of the fuel in the tanks, and (3) small elastic deformations and fuel sloshing. Types of motion (1) and (2) are determined in the basic Six-Degree-of Freedom Flight Path Study Generalized Computer Program (SDF) and the Vehicle Physical Characteristics Subprogram (VPCS). Type (3) is to be determined in the Structural Loads Program (SLP), which, as its name indicates, is also to determine the structural loads.

It is assumed here that the relatively small elastic deformations and fuel sloshing motions have a negligible effect on the other (large) motions of the vehicle. (Some considerations associated with this assumption are investigated in following paragraphs.) It is <u>not</u> assumed that the large motions of the vehicle have a negligible effect on the small motions. Consequently, the large motions - types (1) and (2) - will be employed as part of the input to the Structural Loads Program.

An orthogonal right-handed triad of unit vectors $\bar{J}_1, \bar{J}_2, \bar{J}_3$ that would be fixed in the vehicle if it were perfectly rigid is introduced to provide a frame of reference (a) to represent the gross motion of the vehicle and (b) to facilitate the description of the other motions of the vehicle - types (2) and (3). Rectangular coordinates y^1, y^2, y^3 are associated respectively with the unit vectors $\bar{J}_1, \bar{J}_2, \bar{J}_3$ as shown in Figure 2. These coordinates are seen to be the components of the position vector $\bar{y}$, which equals $\bar{J}_1 y^1 + \bar{J}_2 y^2 + \bar{J}_3 y^3$. The axes of these coordinates are called "vehicle" axes.

Additional orthogonal right-handed triads of unit vectors $\bar{J}_1', \bar{J}_2', \bar{J}_3'$ that would be fixed in the various parts of the vehicle and in the fuel in the various tanks if they were rigid are introduced as frames of reference (a) to represent the motions of the parts and the displacements of the fuel relative to the vehicle and (b) to facilitate the description of the elastic deformations and the sloshing of the fuel. Rectangular coordinates v^1, v^2, v^3 are associated respectively with the unit vectors $\bar{J}_1', \bar{J}_2', \bar{J}_3'$ and these coordinates are the components of the vector $\bar{v}$, which is the position vector in the $\bar{J}_r'$ coordinate system. The axes of these coordinates are called "section" axes. The vector $\bar{\sigma}$ locates the origin of the $\bar{J}_r'$ coordinate system with respect to the $\bar{J}_r$ system. Consequently, then, $\bar{y} = \bar{\sigma} + \bar{v}$.

The gross motion of the vehicle is that of the $\bar{J}_r\ (r = 1, 2, 3)$ triad, which has a linear velocity $\bar{V}$ at its origin and an angular velocity $\bar{\Omega}$. These velocities are functions of the time t, and, together with their derivatives, completely describe the gross motion of the vehicle.

Generalized coordinates are employed to specify the configuration of the vehicle and fuel relative to this frame of reference - the $\bar{J}_r$ triad. In so

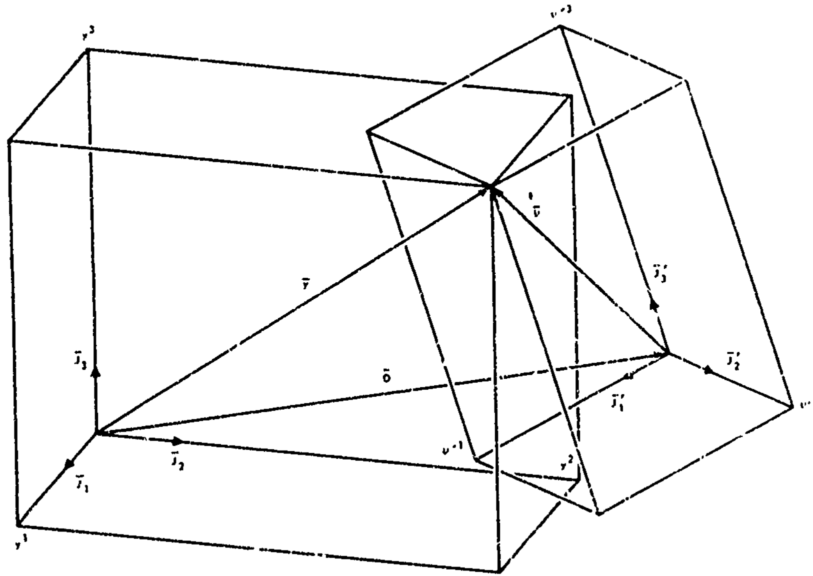


Figure 2. Vehicle and Section Coordinates

doing, however, a distinction is made between the large motions of type (2) and the small motions - type (3). Inasmuch as the large motions are foreknown in the SIP they can be specified by means of a single generalized coordinate, for which the symbol q^0 is chosen here, and which is to be equated to the time t. The use of q^0 for this purpose rather than t, even though the two are numerically equivalent, serves to distinguish the motions of type (2) from the gross motion, type (1), the gross motion being represented as a function of t but not as a function of q^0.

Undamped free vibration modes are used as degrees of freedom for specifying elastic deformations and fuel sloshing, motions of type (3), and are referred to as elastic degrees of freedom. (Good results can reasonably be expected if a sufficient number of the lower frequency modes are used.) These small motions are specified by the generalized coordinates $q^1, q^2, \ldots q^n$ (n being the number of elastic degrees of freedom).

At this point, it is convenient to adopt the range and summation conventions of the tensor analysis as follows:

(1) Range Convention - A coordinate suffix that occurs just once in a term is understood to represent all the integral values appropriate to its range.

(2) Summation Convention - A coordinate suffix that occurs just twice in a term implies summation with respect to that suffix over its range.

These conventions enable us to write

$$\bar{y} = \bar{J}_r\, y^r \left(= \bar{J}_1 y^1 + \bar{J}_2 y^2 + \bar{J}_3 y^3\right) \tag{1}$$

$$\bar{\nu} = \bar{J}'_r\, \nu'^r \left(= \bar{J}'_1 \nu'^1 + \bar{J}'_2 \nu'^2 + \bar{J}'_3 \nu'^3\right) \tag{2}$$

$$\bar{\sigma} = \bar{J}_r\, \sigma^r \left(= \bar{J}_1 \sigma^1 + \bar{J}_2 \sigma^2 + \bar{J}_3 \sigma^3\right) \tag{3}$$

$$\bar{J}'_r = \bar{J}_s\, e^s_r \left(= \bar{J}_1 e^1_r + \bar{J}_2 e^2_r + \bar{J}_3 e^3_r\right) \tag{4}$$

The σ^r in equation (3) are the coordinates in the $\bar{J}_r$ coordinate system of the origin of the $\bar{J}'_r$ system. The e^s_r in (4) are the components in the $\bar{J}_s$ system of the $\bar{J}'_r$ vectors; for any particular choice of r and s, $e^s_r = \bar{J}'_r \cdot \bar{J}_s$, which is the cosine of the angle between $\bar{J}'_r$ and $\bar{J}_s$. Equations (1), (2), and (3) illustrate the use of the summation convention; equation (4) illustrates both conventions.

Unless otherwise noted, the range of the suffix of a unit vector ($\bar{J}_r$ or $\bar{J}'_r$) or of a rectangular coordinate (y^r, ν'^r, or σ^r) is 1, 2, 3. The range of the suffix of a generalized coordinate (q^κ) will be understood to be $1, 2, \ldots n$. Zero (as in q^0) is specifically and deliberately excluded in the use of the range and summation conventions. Thus, in specifying basic functional relations, we write

$$\bar{J}_r = \bar{J}_r(t) \tag{5}$$

$$\sigma^r = \sigma^r(q^0, q^\kappa) \tag{6}$$

$$e^s_r = e^s_r(q^0, q^\kappa) \tag{7}$$

$$\nu'^r = \nu'^r(q^\kappa) \tag{8}$$

Since $\bar{y} = \bar{v} + \bar{c}$, we find by substitution from (1) thru (4) that

$$\bar{J}_r y^r = \bar{J}_r c^r + \bar{I}_r v^r$$
$$= \bar{J}_r c^r + \bar{J}_s e_r^s v^r$$
$$= \bar{J}_r c^r + \bar{J}_r e_s^r v^s \tag{9}$$

$$y^r = c^r + e_s^r v^s = y^r(q^o, q^k) \tag{10}$$

The components y^r of the position vector $\bar{y}$ are the coordinates of a particle of the vehicle, and their variation as the particle moves with respect to the $\bar{J}_r$ frame of reference is a function of q^o and the q^k. Since the angular velocity of the $\bar{J}_r$ triad is $\bar{\Omega}$, and since the $\bar{J}_r$ are functions of the time t only, their derivatives are

$$\frac{d\bar{J}_r}{dt} = \bar{\Omega} \times \bar{J}_r \tag{11}$$

It is clear from (10) that $\frac{\partial y^r}{\partial t} = 0$; therefore, from (1),

$$\frac{\partial \bar{y}}{\partial t} = \frac{d\bar{J}_r}{dt} y^r = \bar{\Omega} \times \bar{J}_r y^r = \bar{\Omega} \times \bar{y} \tag{12}$$

and

$$\frac{d\bar{y}}{dt} = \frac{\partial \bar{y}}{\partial t} + \frac{\partial \bar{y}}{\partial q^o} \dot{q}^o + \frac{\partial \bar{y}}{\partial q^k} \dot{q}^k$$

$$= \bar{\Omega} \times \bar{y} + \frac{\partial \bar{y}}{\partial q^o} + \frac{\partial \bar{y}}{\partial q^k} \dot{q}^k , \tag{13}$$

since $\dot{q}^o = \dfrac{dq^o}{dt} = 1.$ (14)

$\dot{q}^{\iota} = \frac{dq^{\iota}}{dt}$ and is unknown until determined in the solution of the equations of motion, which are to follow. Making use of (13) and the fact that the linear velocity of the origin of the $\bar{J}_r$ triad is $\bar{V}$, we find that the velocity of a particle of the vehicle is

$$\bar{v} = \bar{V} + \frac{d\bar{y}}{dt}$$

$$= \bar{V} + \bar{\Omega} \times \bar{y} + \frac{\partial \bar{y}}{\partial q^{\circ}} + \frac{\partial \bar{y}}{\partial q^{\kappa}} \dot{q}^{\kappa} \tag{15}$$

The acceleration of a particle can be found as follows

$$\frac{\partial}{\partial t}\left(\frac{\partial \bar{y}}{\partial q^{\circ}}\right) = \frac{d\bar{J}_r}{dt} \frac{\partial y^r}{\partial q^{\circ}} = \bar{\Omega} \times \bar{J}_r \frac{\partial y^r}{\partial q^{\circ}} = \bar{\Omega} \times \frac{\partial \bar{y}}{\partial q^{\circ}} . \tag{16}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{y}}{\partial q^{\circ}}\right) = \bar{\Omega} \times \frac{\partial \bar{y}}{\partial q^{\circ}} + \frac{\partial^2 \bar{y}}{\partial q^{\circ} \partial q^{\circ}} + \frac{\partial^2 \bar{y}}{\partial q^{\circ} \partial q^{\kappa}} \dot{q}^{\kappa} . \tag{17}$$

Likewise

$$\frac{d}{dt}\left(\frac{\partial \bar{y}}{\partial q^{\kappa}}\right) = \bar{\Omega} \times \frac{\partial \bar{y}}{\partial q^{\kappa}} + \frac{\partial^2 \bar{y}}{\partial q^{\kappa} \partial q^{\circ}} + \frac{\partial^2 \bar{y}}{\partial q^{\kappa} \partial q^{\iota}} \dot{q}^{\iota} . \tag{18}$$

The acceleration is now found by differentiation of (15) and substitution from (13), (17), and (18), with the result

$$\frac{d\bar{v}}{dt} = \frac{d\bar{V}}{dt} + \bar{\Omega} \times \frac{d\bar{y}}{dt} + \frac{d\bar{\Omega}}{dt} \times \bar{y} + \frac{d}{dt}\left(\frac{\partial \bar{y}}{\partial q^{\circ}}\right) + \frac{d}{dt}\left(\frac{\partial \bar{y}}{\partial q^{\kappa}}\right)\dot{q}^{\kappa} + \frac{\partial \bar{y}}{\partial q^{\kappa}} \ddot{q}^{\kappa}$$

$$= \frac{d\bar{V}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}) + \frac{d\bar{\Omega}}{dt} \times \bar{y} + 2\,\bar{\Omega} \times \frac{\partial \bar{y}}{\partial q^{\circ}} + \frac{\partial^2 \bar{y}}{\partial q^{\circ} \partial q^{\circ}}$$

$$+ 2\left(\bar{\Omega} \times \frac{\partial \bar{y}}{\partial q^{\kappa}} + \frac{\partial^2 \bar{y}}{\partial q^{\circ} \partial q^{\kappa}}\right)\dot{q}^{\kappa} + \frac{\partial^2 \bar{y}}{\partial q^{\kappa} \partial q^{\iota}} \dot{q}^{\kappa} \dot{q}^{\iota} + \frac{\partial \bar{y}}{\partial q^{\kappa}} \ddot{q}^{\kappa} \tag{19}$$

We note that

$$\frac{d\bar{\Omega}}{dt} = \bar{J}_r \frac{d\Omega^r}{dt} + \frac{d\bar{J}_r}{dt}\Omega^r$$

$$= \bar{J}_r \dot{\Omega}^r + \bar{\Omega} \times \bar{J}_r \Omega^r$$

$$= \bar{J}_r \dot{\Omega}^r, \tag{20}$$

because $\bar{\Omega} \times \bar{J}_r \Omega^r = \bar{\Omega} \times \bar{\Omega} = 0.$

The $\dot{\Omega}^r$, then, are the components of the angular acceleration. The linear acceleration at the origin is

$$\frac{d\bar{V}}{dt} = \bar{J}_r \frac{dV^r}{dt} + \frac{d\bar{J}_r}{dt} V^r$$

$$= \bar{J}_r \dot{V}^r + \bar{\Omega} \times \bar{V}$$

$$= \bar{J}_1 \left(\dot{V}^1 + \Omega^2 V^3 - \Omega^3 V^2\right)$$

$$+ \bar{J}_2 \left(\dot{V}^2 + \Omega^3 V^1 - \Omega^1 V^3\right)$$

$$+ \bar{J}_3 \left(\dot{V}^3 + \Omega^1 V^2 - \Omega^2 V^1\right) \tag{21}$$

The coefficients of the unit vectors are the components of the linear acceleration.

3. FORCES, MOMENTS, AND "DYNAMIC BALANCING"

In the application of Newton's second law of motion to the vehicle, it is necessary to have a means of identifying the particles. But the vehicle is divided into various parts (or sections) and various fuel tanks, which also need to be identified. Because of its shifty and sloshy nature, the fuel cannot be regarded as part of the tank that contains it. The tank itself is treated as one or more structural sections. Subscripts are employed to identify masses or mass particles and their rectangular coordinates, a single subscript or the first of two subscripts denoting the section or the fuel contained in a certain tank, and the second subscript denoting the particle of the section or fuel. The absence of such subscripts denotes a quantity pertaining to the entire vehicle.

Thus the mass of the h-th particle of the i-th section is m_{ih} and its coordinates are y^r_{ih} and ν^r_{ih}. The mass of the i-th section is m_i and its "coordinates" are θ^r_i. The mass of the entire vehicle is m (without a subscript). Let P_i be the number of particles in the i-th section or tank of fuel and N be the number of sections and tanks,

$$\text{then} \quad m_i = \sum_{h=1}^{P_i} m_{ih} \tag{22}$$

$$\text{and} \quad m = \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} = \sum_{i=1}^{N} m_i \tag{23}$$

Let the $\bar{J}'_r$ triad of the i-th section or tank of fuel be designated as the $\bar{J}'_{ri}$ triad and let its origin be at the center of mass of the i-th system of particles. Then the θ^r_i are the coordinates of the center of mass of the i-th section, and

$$\sum_{h=1}^{P_i} m_{ih} \, \nu^r_{ih} = 0 \tag{24}$$

Also, let y^r_c be the coordinates of the center of mass of the vehicle. Then, with the aid of (10), (22), and (24) it is found that

$$\begin{aligned} m y^r_c &= \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \, y^r_{ih} \\ &= \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \left(c^r_i + e^r_{si} \nu'^s_{ih} \right) \\ &= \sum_{i=1}^{N} m_i \, \theta^r_i \end{aligned} \tag{25}$$

[illegible] in vector notation

[illegible]

$\bar{y}_0$ [illegible] position vector of the center of mass of the vehicle

With the aid of Newton's third law of motion, it can be shown that the [illegible] forces exerted on all the particles of the vehicle equals the sum of the external forces only. Let this be designated by $\bar{F}$, then [illegible] second law of motion and (17),

$$\bar{F} = \sum_{i=1}^{N} \sum_{h=1}^{r} m_{ih} \frac{d\bar{v}_{ih}}{dt}$$

$$= \sum_{i=1}^{N} \sum_{h=1}^{r} m_{ih} \Big[\frac{d\bar{V}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{ih}) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{ih}$$

$$+ 2\bar{\Omega} \times \frac{\partial \bar{y}_{ih}}{\partial q^i} + \frac{\partial^2 \bar{y}_{ih}}{\partial q^i \partial q^i} + 2\left(\bar{\Omega} \times \frac{\partial \bar{y}_{ih}}{\partial q^k} + \frac{\partial^2 \bar{y}_{ih}}{\partial q^i \partial q^k}\right) \dot{q}^k$$

$$+ \frac{\partial^2 \bar{y}_{ih}}{\partial q^k \partial q^l} \dot{q}^k \dot{q}^l + \frac{\partial \bar{y}_{ih}}{\partial q^k} \ddot{q}^k \Big] \tag{27}$$

Likewise, the sum of the moments about the origin of the $\bar{J}_r$ triad due to all the forces is equal to the sum of the moments due to the external forces only. Let this be designated by $\bar{G}$, then, by Newton's second law of motion and (19),

$$\bar{G} = \sum_{i=1}^{N} \sum_{h=1}^{r} \bar{y}_{ih} \times \left(m_{ih} \frac{d\bar{v}_{ih}}{dt}\right)$$

$$= \sum_{i=1}^{N} \sum_{h=1}^{r} m_{ih}\, \bar{y}_{ih} \times \Big[\frac{d\bar{V}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{ih}) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{ih}$$

$$+ 2\bar{\Omega} \times \frac{\partial \bar{y}_{ih}}{\partial q^i} + \frac{\partial^2 \bar{y}_{ih}}{\partial q^i \partial q^i} + 2\left(\bar{\Omega} \times \frac{\partial \bar{y}_{ih}}{\partial q^k} + \frac{\partial^2 \bar{y}_{ih}}{\partial q^i \partial q^k}\right) \dot{q}^k$$

$$+ \frac{\partial^2 \bar{y}_{ih}}{\partial q^k \partial q^l} \dot{q}^k \dot{q}^l + \frac{\partial \bar{y}_{ih}}{\partial q^k} \ddot{q}^k \Big] \tag{28}$$

Inasmuch as the J_r triad provides a frame of reference to represent the gross motion of the vehicle, its linear and angular velocities and accelerations $\bar{V}, \bar{\Omega}, d\bar{V}/dt$, and $d\bar{\Omega}/dt$ are those of the vehicle as a whole. It would be strictly proper to require these velocities and accelerations to satisfy (27) and (28); but it has been assumed that the elastic deformations and fuel sloshing, motions of type (3), have a negligible effect on the large motions of the vehicle; therefore, $\bar{V}, \bar{\Omega}$, $d\bar{V}/dt$, and $d\bar{\Omega}/dt$ are regarded as not being functions of the generalized coordinates q or their derivatives $\dot{q}$ and $\ddot{q}$. The fact that $\bar{F}$ and $\bar{G}$ may be significantly affected by the elastic deformation of aerodynamic surfaces is arbitrarily disregarded here, and the portions of (27) and (28) involving $\dot{q}^\kappa$ and $\ddot{q}^\kappa$ are simply ignored in the process of determining the gross motion of the vehicle. This leaves us, for the determination of the gross motion,

$$\bar{F} = \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih}\left[\frac{d\bar{V}}{dt} + \bar{\Omega} \times \left(\bar{\Omega} \times \bar{y}_{ih}\right) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{ih} + 2\bar{\Omega} \times \frac{\partial \bar{y}_{ih}}{\partial q^\tau} + \frac{\partial^2 \bar{y}_{ih}}{\partial q^\sigma \partial q^\tau}\right] \tag{29}$$

$$\bar{G} = \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih}\, \bar{y}_{ih} \times \left[\frac{d\bar{V}}{dt} + \bar{\Omega} \times \left(\bar{\Omega} \times \bar{y}_{ih}\right) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{ih} + 2\bar{\Omega} \times \frac{\partial \bar{y}_{ih}}{\partial q^\tau} + \frac{\partial^2 \bar{y}_{ih}}{\partial q^\sigma \partial q^\tau}\right] \tag{30}$$

with none of these terms being regarded as functions of the q^κ

Equation (29) contains the summations $\sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih}\, \bar{y}_{ih}$ and $\sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^\tau}$. If the terms of (29) are not to be functions of the q^κ, the partial derivatives of these summations with respect to the q^κ should be equal to zero. Furthermore, if these same partial derivatives are equal to zero, the terms of (27) involving the $\dot{q}^\kappa$ and the $\ddot{q}^\kappa$ will vanish, because they contain these partial derivatives as factors. In fact, it is sufficient for this purpose for

to be zero, because $\sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^\kappa}$

$$\frac{\partial}{\partial q^\tau} \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^\sigma} = \frac{\partial}{\partial q^\sigma} \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^\kappa} = 0$$

if $\sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^\kappa} = 0.$

Reference to Equation (26) sheds a little more light on this problem:

$$\sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^\kappa} = \sum_{i=1}^{N} m_i \frac{\partial \bar{\delta}_i}{\partial q^\kappa} = m \frac{\partial \bar{y}_c}{\partial q^\kappa} \tag{31}$$

From the physical viewpoint it is clear that $\bar{y}_c$ will not change much as a result of elastic deformation, but that fact does not preclude the possibility of its changing rapidly, therefore, it is rash to assume arbitrarily that (31) will equal zero. Rather, it is desirable to impose its being zero as a condition to be satisfied by the elastic degrees of freedom.

The components of the inertia tensor (or elements of the inertia matrix) of the vehicle are given by the formula (Reference 7, Equation 2-12)

$$M_{jk} = \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^j} \cdot \frac{\partial \bar{y}_{ih}}{\partial q^k} \tag{32}$$

It is known that $M_{jk} = 0$ when the j-th degree of freedom is a form of motion in which the vehicle moves as a rigid body, having a translation rate $\bar{c}_j$ and a rotation rate $\bar{b}_j$ relative to the $\bar{J}_r$ triad, and the k-th degree of freedom is a normal free-free mode of vibration. When such is the case,

$$\frac{\partial \bar{y}_{ih}}{\partial q^j} = \bar{c}_j + \bar{b}_j \times \bar{y}_{ih} \tag{33}$$

and from (32)

$$\begin{aligned} M_{jk} &= \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \left(\bar{c}_j + \bar{b}_j \times \bar{y}_{ih} \right) \cdot \frac{\partial \bar{y}_{ih}}{\partial q^k} \\ &= \bar{c}_j \cdot \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^k} + \bar{b}_j \cdot \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \bar{y}_{ih} \times \frac{\partial \bar{y}_{ih}}{\partial q^k} \\ &= 0 \end{aligned} \tag{34}$$

Now $\bar{c}_j$ and $\bar{b}_j$, being __any__ translation and rotation rates of the vehicle, are arbitrary; therefore,

$$\sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^k} = 0 \tag{35}$$

and

$$\sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} \bar{y}_{ih} \times \frac{\partial \bar{y}_{ih}}{\partial q^k} = 0 \tag{36}$$

when the k-th degree of freedom is a normal free-free mode of vibration.

Thus, if the elastic degrees of freedom satisfy the condition that they are normal free-free modes of vibration, Equation (35) is satisfied, the terms in (27) involving G' and G^{K} vanish (eliminating the difference between (27) and (29)), and the terms on the right side of (29) are not functions of the q^{K}. However, the use of normal free-free modes accomplishes more than this. Equation (36) directly eliminates one term of (28), and differentiation of (36) leads to the elimination of other terms, as follows

$$\frac{\partial}{\partial q^{l}} \sum_{i=1}^{N} \sum_{h=1}^{P} m_{ih}\, \bar{y}_{ih} \times \frac{\partial \bar{y}_{ih}}{\partial q^{k}}$$

$$= \sum_{i=1}^{N} \sum_{h=1}^{P} m_{ih}\, \bar{y}_{ih} \times \frac{\partial^2 \bar{y}_{ih}}{\partial q^{k} \partial q^{l}} + \sum_{i=1}^{N} \sum_{h=1}^{P} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^{l}} \times \frac{\partial \bar{y}_{ih}}{\partial q^{k}}$$

$$= 0, \text{ or} \tag{37}$$

$$\sum_{i=1}^{N} \sum_{h=1}^{P} m_{ih}\, \bar{y}_{ih} \times \frac{\partial^2 \bar{y}_{ih}}{\partial q^{k} \partial q^{l}} = \sum_{i=1}^{N} \sum_{h=1}^{P} m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^{k}} \times \frac{\partial \bar{y}_{ih}}{\partial q^{l}}$$

$$= 0 \tag{38}$$

because interchanging the superscripts k and l does not affect the left side of (38) whereas it reverses the sign of the right side, and only zero equals its opposite. Furthermore, the superscript l could be replaced by 0 in the two equations above; thus, (28) is reduced to

$$\bar{G} = \sum_{i=1}^{N} \sum_{h=1}^{P} m_{ih}\, \bar{y}_{ih} \times \left[\frac{d\bar{V}}{dt} + \bar{\Omega} \times \left(\bar{\Omega} \times \bar{y}_{ih}\right) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{ih}\right.$$

$$\left. + 2\,\bar{\Omega} \times \frac{\partial \bar{y}_{ih}}{\partial q^{0}} + \frac{\partial^2 \bar{y}_{ih}}{\partial q^{0} \partial q^{0}} + 2\,\bar{\Omega} \times \frac{\partial \bar{y}_{ih}}{\partial q^{k}} \dot{q}^{k}\right] \tag{39}$$

which could be used for whatever value it might have in solving for the $\dot{q}^{K}$, it being recognized that the terms of this equation <u>are</u> functions of the q^{K}, in contrast to the use of equations (29) and (30).

The results thus accomplished by the use of normal free-free modes of vibration can also be brought about by a "dynamic balancing" of each degree of freedom individually. In order to do this, let

$$\frac{\partial \bar{y}_{ih}}{\partial q^{k}} = \bar{a}_{k,ih} + \bar{b}_{k} \times \bar{y}_{ih} + \bar{c}_{k} \tag{40}$$

the $\bar{a}_{kih}$ being given (not necessarily free-free) modes of vibration, and the $\bar{b}_k$ and $\bar{c}_k$ being as defined in connection with (33) except that, instead of being arbitrary, they are now unknowns to be determined in such a way that (35) and (36) will be satisfied. Substitution from (40) into (35) results in

$$\sum_{i=1}^{\nu} \sum_{h=1}^{n} m_{ih} \left(\bar{a}_{kih} + \bar{b}_k \times \bar{y}_{ih} + \bar{c}_k\right)$$

$$= \sum_{i=1}^{\nu} \sum_{h=1}^{n} m_{ih} \bar{a}_{kih} + m \bar{b}_k \times \bar{y}_c + m \bar{c}_k = 0, \tag{41}$$

and substitution into (36) results in

$$\sum_{i=1}^{\nu} \sum_{h=1}^{n} m_{ih} \bar{y}_{ih} \times \left(\bar{a}_{kih} + \bar{b}_k \times \bar{y}_{ih} + \bar{c}_k\right)$$

$$= \sum_{i=1}^{\nu} \sum_{h=1}^{n} m_{ih} \bar{y}_{ih} \times \left(\bar{a}_{kih} + \bar{b}_k \times \bar{y}_{ih}\right) + m \bar{y}_c \times \bar{c}_k$$

$$= 0 \tag{42}$$

Now let us eliminate $\bar{c}_k$ by forming the vector product of $\bar{y}_c$ with (41) and subtracting it from (42). This results in

$$\sum_{i=1}^{\nu} \sum_{h=1}^{n} m_{ih} \bar{y}_{ih} \times \left(\bar{a}_{kih} + \bar{b}_k \times \bar{y}_{ih}\right)$$

$$- \bar{y}_c \times \sum_{i=1}^{\nu} \sum_{h=1}^{n} m_{ih} \bar{a}_{kih} - m \bar{y}_c \times \left(\bar{b}_k \times \bar{y}_c\right) = 0, \tag{43}$$

which can be solved for the $\bar{b}_k$. Once the $\bar{b}_k$ are obtained, (41) can be used to obtain the $\bar{c}_k$. When the $\bar{b}_k$ and $\bar{c}_k$ are obtained in this manner, the use of (40) results in $\frac{\partial \bar{y}_{ih}}{\partial \xi_k}$ that satisfy (35) and (36). These may be called "dynamically balanced" modes. They have the practical advantage of being much more easily obtained than the normal free-free modes.

4. EQUATIONS OF MOTION FOR THE ELASTIC DEFORMATIONS

In the preceding section, the influence of the internal forces and the distribution of the aerodynamic pressures over the surface of the vehicle were deliberately disregarded. In this section, it will be necessary to give them full consideration, because their effect on the elastic deformations cannot be disregarded and because the purpose of this section is to deduce equations of motion for the determination of the elastic deformations in the various degrees of freedom. For the sake of suitable notation, let $\bar{F}_{ih}$ denote the external force on the h-th particle of the i-th section, and let $\bar{F}_{ihkj}$ represent the internal force exerted on the h-th particle of the i-th section by the j-th particle of the k-th section. By Newton's second law of motion, then, the total force exerted against the h-th particle of the i-th section is

$$m_{ih}\frac{d\bar{v}_{ih}}{dt} = \bar{F}_{ih} + \sum_{k=1}^{N}\sum_{j=1}^{p_k}\bar{F}_{ihkj} \quad . \tag{44}$$

The equations of motion in terms of generalized forces are obtained from (44) by forming the scalar product of $\frac{\partial\bar{r}_{ih}}{\partial q^j}$ with each term and summing over h and i. Thus

$$\sum_{i=1}^{N}\sum_{h=1}^{p_i} m_{ih}\frac{\partial\bar{r}_{ih}}{\partial q^j}\cdot\frac{d\bar{v}_{ih}}{dt} = \sum_{i=1}^{N}\sum_{h=1}^{p_i}\frac{\partial\bar{r}_{ih}}{\partial q^j}\cdot\bar{F}_{ih}$$

$$+\sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k}\frac{\partial\bar{r}_{ih}}{\partial q^j}\cdot\bar{F}_{ihkj} \quad . \tag{45}$$

For convenience, the generalized forces are separated into four types and designated as follows:

1. Those associated with inertia forces are

$$N_j = \sum_{i=1}^{N}\sum_{h=1}^{p_i} m_{ih}\frac{\partial\bar{r}_{ih}}{\partial q^j}\cdot\frac{d\bar{v}_{ih}}{dt} \quad . \tag{46}$$

2. Those associated with external forces are

$$Q_j = \sum_{i=1}^{N}\sum_{h=1}^{p_i}\frac{\partial\bar{r}_{ih}}{\partial q^j}\cdot\bar{F}_{ih} \quad . \tag{47}$$

3. Those associated with conservative internal forces are O_j.

4. Those associated with dissipative internal forces are P_j.

Since the $\bar{F}_{ihkj}$ denote the internal forces, we may let

$$O_j + P_j = -\sum_{i=1}^{N}\sum_{k=1}^{P_i}\sum_{h=1}^{N}\sum_{g=1}^{P_h} \frac{\partial \bar{y}_{ik}}{\partial q^j} \cdot \bar{F}_{ikhg} \ , \tag{48}$$

and substitution from these last three equations into (45) leads to

$$N_j - O_j + P_j = Q_j \ . \tag{49}$$

Substitution from (19) into (46) results in

$$\begin{aligned} N_j = \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \frac{\partial \bar{y}_{ik}}{\partial q^j} \cdot \Big[& \frac{d\bar{V}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{ik}) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{ik} \\ & + 2\bar{\Omega} \times \frac{\partial \bar{y}_{ik}}{\partial q^0} + \frac{\partial^2 \bar{y}_{ik}}{\partial q^0 \partial q^0} + 2\Big(\bar{\Omega} \times \frac{\partial \bar{y}_{ik}}{\partial q^k} + \frac{\partial^2 \bar{y}_{ik}}{\partial q^0 \partial q^k}\Big)\dot{q}^k \\ & + \frac{\partial^2 \bar{y}_{ik}}{\partial q^k \partial q^l}\dot{q}^k \dot{q}^l + \frac{\partial \bar{y}_{ik}}{\partial q^k}\ddot{q}^k \Big] . \end{aligned} \tag{50}$$

If we make use of (32), (35), (36), (38), and some new symbols in an examination of the individual terms of (50), we obtain a simpler and more practical expression for N_j, as follows:

$$\sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \frac{\partial \bar{y}_{ik}}{\partial q^j} \cdot \frac{d\bar{V}}{dt} = \frac{d\bar{V}}{dt} \cdot \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \frac{\partial \bar{y}_{ik}}{\partial q^j} = 0 \tag{51}$$

$$\begin{aligned} & \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \frac{\partial \bar{y}_{ik}}{\partial q^j} \cdot \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{ik}) \\ & = \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \Big[\Big(\bar{\Omega} \cdot \frac{\partial \bar{y}_{ik}}{\partial q^j}\Big)(\bar{\Omega} \cdot \bar{y}_{ik}) - \Big(\frac{\partial \bar{y}_{ik}}{\partial q^j} \cdot \bar{y}_{ik}\Big)(\bar{\Omega} \cdot \bar{\Omega})\Big] \\ & = \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \Big[\Omega^r y^r_{ik} \Omega^s \frac{\partial y^s_{ik}}{\partial q^j} - \Omega^r \Omega^r y^t_{ik} \frac{\partial y^t_{ik}}{\partial q^j}\Big] \\ & = \Omega^r \Omega^s \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \Big(y^r_{ik} \frac{\partial y^s_{ik}}{\partial q^j} - \delta_{rs} y^t_{ik} \frac{\partial y^t_{ik}}{\partial q^j}\Big) = -\Omega^r \Omega^s P_{rsj} \end{aligned} \tag{52}$$

where

$$P_{rsj} = \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \Big(\delta_{rs} y^t_{ik} \frac{\partial y^t_{ik}}{\partial q^j} - y^r_{ik} \frac{\partial y^s_{ik}}{\partial q^j}\Big) \ . \tag{53}$$

$$\begin{aligned} \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \frac{\partial \bar{y}_{ik}}{\partial q^j} \cdot \frac{d\bar{\Omega}}{dt} \times \bar{y}_{ik} & = \frac{d\bar{\Omega}}{dt} \cdot \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik} \bar{y}_{ik} \times \frac{\partial \bar{y}_{ik}}{\partial q^j} \\ & = 0 \ . \end{aligned} \tag{54}$$

$$\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\frac{\partial \bar{y}_{ih}}{\partial q^j}\cdot\bar{\Omega}\times\frac{\partial \bar{y}_{ih}}{\partial q^0}=\bar{\Omega}\cdot\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\frac{\partial \bar{y}_{ih}}{\partial q^0}\times\frac{\partial \bar{y}_{ih}}{\partial q^j}=0$$

$$\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\frac{\partial \bar{y}_{ih}}{\partial q^j}\cdot\frac{\partial^2 \bar{y}_{ih}}{\partial q^0\partial q^0}=\boxed{00,j} \tag{56}$$

$$\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\frac{\partial \bar{y}_{ih}}{\partial q^j}\cdot\bar{\Omega}\times\frac{\partial \bar{y}_{ih}}{\partial q^k}=\bar{\Omega}\cdot\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\frac{\partial \bar{y}_{ih}}{\partial q^k}\times\frac{\partial \bar{y}_{ih}}{\partial q^j}$$
$$=0 \tag{57}$$

$$\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\frac{\partial \bar{y}_{ih}}{\partial q^j}\cdot\frac{\partial^2 \bar{y}_{ih}}{\partial q^0\partial q^k}=\boxed{0K,j} \tag{58}$$

$$\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\frac{\partial \bar{y}_{ih}}{\partial q^j}\cdot\frac{\partial^2 \bar{y}_{ih}}{\partial q^k\partial q^l}=\boxed{KL,j} \tag{59}$$

$$\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\frac{\partial \bar{y}_{ih}}{\partial q^j}\cdot\frac{\partial \bar{y}_{ih}}{\partial q^k}=M_{jk}\,. \tag{60}$$

Substitution from (51) through (60) into (50) results in

$$N_j=-\Omega^r\Omega^s P_{rsj}+\boxed{00,j}+2\boxed{0K,j}\,\dot{q}^k$$
$$+\boxed{KL,j}\,\dot{q}^k\dot{q}^l+M_{jk}\ddot{q}^k. \tag{61}$$

It is also possible to use the familiar Lagrangean expression for N_j in terms of the kinetic energy T. To show that this is so, we note from (15) that

$$\frac{\partial \bar{v}_{ih}}{\partial \dot{q}^k}=\frac{\partial \bar{y}_{ih}}{\partial q^k}. \tag{62}$$

and from (15) and (18) that

$$\frac{\partial \bar{v}_{ih}}{\partial q^j}=\bar{\Omega}\times\frac{\partial \bar{y}_{ih}}{\partial q^j}+\frac{\partial^2 \bar{y}_{ih}}{\partial q^0\partial q^j}+\frac{\partial^2 \bar{y}_{ih}}{\partial q^j\partial q^k}\dot{q}^k$$
$$=\frac{d}{dt}\left(\frac{\partial \bar{y}_{ih}}{\partial q^j}\right). \tag{63}$$

The kinetic energy is given by the well known formula

$$T=\frac{1}{2}\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\,\bar{v}_{ih}\cdot\bar{v}_{ih}\,, \tag{64}$$

whence, with the aid of (62) and (63),

$$\frac{\partial T}{\partial \dot{q}^j}=\sum_{i=1}^{\nu}\sum_{h=1}^{\rho_i} m_{ih}\,\bar{v}_{ih}\cdot\frac{\partial \bar{v}_{ih}}{\partial \dot{q}^j}$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{P_i} m_{ik}\, \bar{v}_{ik} \cdot \frac{\partial \bar{y}_{ik}}{\partial q^j} , \tag{65}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^j}\right) = \sum_{i=1}^{N} \sum_{k=1}^{P_i} m_{ik} \left[\frac{d\bar{v}_{ik}}{dt} \cdot \frac{\partial \bar{y}_{ik}}{\partial q^j} + \bar{v}_{ik} \cdot \frac{d}{dt}\left(\frac{\partial \bar{y}_{ik}}{\partial q^j}\right)\right] , \tag{66}$$

and

$$\frac{\partial T}{\partial q^j} = \sum_{i=1}^{N} \sum_{k=1}^{P_i} m_{ik}\, \bar{v}_{ik} \cdot \frac{\partial \bar{v}_{ik}}{\partial q^j}$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{P_i} m_{ik}\, \bar{v}_{ik} \cdot \frac{d}{dt}\left(\frac{\partial \bar{y}_{ik}}{\partial q^j}\right). \tag{67}$$

Subtraction of (67) from (66) results in the Lagrangian expression

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^j}\right) - \frac{\partial T}{\partial q^j} = \sum_{i=1}^{N} \sum_{k=1}^{P_i} m_{ik} \frac{\partial \bar{y}_{ik}}{\partial q^j} \cdot \frac{d\bar{v}_{ik}}{dt} = N_j \tag{68}$$

by the defining equation (46). The use of this expression to obtain (61) leads to the interesting discovery that

$$\Omega^r \Omega^s P_{rsj} = \tfrac{1}{2} \Omega^r \Omega^s \frac{\partial I_{rs}}{\partial q^j} \tag{69}$$

or that $\dfrac{\partial I_{rs}}{\partial q^j} = P_{rsj} + P_{srj}$, (70)

where $$I_{rs} = \sum_{i=1}^{N} \sum_{k=1}^{P_i} m_{ik} \left(\delta_{rs}\, y^t_{ik} y^t_{ik} - y^r_{ik} y^s_{ik} \right) \tag{71}$$

= moments and negatives of products of inertia of structure and fuel about vehicle axes.

It is also interesting and useful to observe from (60), (62), (65), and the fact that $\partial \bar{y}_{ik}/\partial q^j$ is not a function of the $\dot{q}^K$ that

$$\overset{**}{m}_{jK} = \frac{\partial^2 T}{\partial \dot{q}^j \partial \dot{q}^K} . \tag{72}$$

This is especially useful in treating the inertia effects of fuel slosh.

Recalling the definition of $\bar{F}_{ikhj}$, we know by Newton's third law of motion that

$$\bar{F}_{ikhj} = -\bar{F}_{hjik} , \tag{73}$$

and that $\bar{F}_{ikhj}$ is parallel to $\bar{y}_{ik} - \bar{y}_{hj}$,

so that $\overline{F}_{ihkj} \times (\overline{y}_{ih} - \overline{y}_{kj}) = 0$ (74)

Substitution from (73) into (48) results in

$$
\begin{aligned}
O_j + P_j - \sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} \frac{\partial \overline{y}_{ih}}{\partial q^i} \cdot \overline{F}_{kjih} \\
= \sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} \frac{\partial \overline{y}_{kj}}{\partial q^i} \cdot \overline{F}_{ihkj} .
\end{aligned}
\tag{75}
$$

If (45) is added to (75) and the sum divided by 2, the result is

$$
O_j + P_j = -\frac{1}{2}\sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} \left[\frac{\partial}{\partial q^i}(\overline{y}_{ih} - \overline{y}_{kj})\right] \cdot \overline{F}_{ihkj} \tag{76}
$$

Let $|\overline{y}_{ih} - \overline{y}_{kj}| = \Delta_{ihkj}$, (77)

which is the distance from particle kj to particle ih; and

let $|\overline{F}_{ihkj}| = F_{ihkj}$, (78)

positive when it tends to increase Δ_{ihkj} and negative when it tends to decrease Δ_{ihkj}.

Then, since $\overline{F}_{ihkj}$ is parallel to $\overline{y}_{ih} - \overline{y}_{kj}$, it can be shown that (76) is equivalent to

$$
O_j + P_j = -\frac{1}{2}\sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} F_{ihkj}\,\frac{\partial}{\partial q^i}\,\Delta_{ihkj} . \tag{79}
$$

Let U be the potential energy due to elastic deformation, and let V be the energy dissipated through damping. These both represent work done in overcoming internal forces; therefore,

$$
\begin{aligned}
\frac{d}{dt}(U+V) &= -\sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} \overline{F}_{ihkj} \cdot \overline{v}_{ih} \\
&= \sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} \overline{F}_{kjih} \cdot \overline{v}_{ih} \\
&= \sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} \overline{F}_{ihkj} \cdot \overline{v}_{kj} \\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} \overline{F}_{ihkj} \cdot (\overline{v}_{ih} - \overline{v}_{kj}) \\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{h=1}^{p_i}\sum_{k=1}^{N}\sum_{j=1}^{p_k} \overline{F}_{ihkj} \cdot \Big[\overline{\Omega} \times (\overline{y}_{ih} - \overline{y}_{kj}) \\
&\qquad + \frac{\partial}{\partial q^0}(\overline{y}_{ih} - \overline{y}_{kj}) + \frac{\partial}{\partial q^i}(\overline{y}_{ih} - \overline{y}_{kj})\,\dot{q}^i\Big]
\end{aligned}
\tag{80}
$$

because of (15). The term

$$\bar{F}_{ikhj} \, \bar{\Omega} \times (\bar{y}_{ik} - \bar{y}_{hj}) = - \bar{\Omega} \, \bar{F}_{ikhj} \times (\bar{y}_{ik} - \bar{y}_{hj}) = 0 \tag{81}$$

because of (14); therefore, because of (76),

$$\frac{d}{dt}(U+V) = O_o + P_o + (O_j + P_j)\dot{q}^j \; , \tag{82}$$

where $O_o + P_o =$

$$-\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{n}\sum_{h=1}^{N}\sum_{j=1}^{n} \bar{F}_{ikhj} \cdot \frac{\partial}{\partial q^o}(\bar{y}_{ik} - \bar{y}_{hj})$$

$$= -\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{n}\sum_{h=1}^{N}\sum_{j=1}^{n} F_{ikhj} \frac{\partial}{\partial q^o} \Delta_{ikhj} \; . \tag{83}$$

Since the O_j are associated with conservative internal forces and the P_j are associated with dissipative internal forces, it is clear from the definitions of U and V and from (82) that

$$\frac{dU}{dt} = O_o + O_j \dot{q}^j \tag{84}$$

and

$$\frac{dV}{dt} = P_o + P_j \dot{q}^j \; . \tag{85}$$

Now U is a function of q^o and the q^j but not of their time derivatives; therefore,

$$\frac{dU}{dt} = \frac{\partial U}{\partial q^o} + \frac{\partial U}{\partial q^j}\dot{q}^j \; . \tag{86}$$

From (84) and (86), we see that

$$O_o = \frac{\partial U}{\partial q^o} \quad \text{and} \quad O_j = \frac{\partial U}{\partial q^j} \; . \tag{87}$$

This can be extremely helpful in the computation of the O_j.

If the F_{ikhj} are only the dissipative internal forces, equation (79) may be directly useful for the calculation of the P_j. In using equation (47) to compute the Q_j, it is not necessary to include the force due to gravity because such a force can be represented as $m_{ik}\bar{g}$, in which case

$$\sum_{i=1}^{N} \sum_{k=1}^{P_i} m_{ik} \bar{g} \cdot \frac{\partial \bar{y}_{ik}}{\partial q^j} = \bar{g} \cdot \sum_{i=1}^{N} \sum_{k=1}^{P_i} m_{ik} \frac{\partial \bar{y}_{ik}}{\partial q^j} = 0 \tag{88}$$

when (35) is satisfied.

Let us introduce certain assumptions here and proceed to some further treatment of the O_j and P_j. First, let us assume that U is a minimum when the q^j equal zero; then

$$O_j = \frac{\partial U}{\partial q^j} = 0 \quad \text{when the } q^j = 0 \; ; \tag{89}$$

and, as a close and convenient approximation (Reference 7, Equation 4-44)

$$O_j = K_{jK} q^K \, , \tag{90}$$

where $K_{jK} = \frac{\partial^2 U}{\partial q^j \partial q^K}$ evaluated for the $q^j = 0$.

Inasmuch as undamped free vibration modes are used as degrees of freedom for specifying elastic deformations and fuel sloshing, there is a frequency ω_j associated with the j-th degree of freedom for all the values of j. For the first degree of freedom,

$$\omega_1 = \sqrt{\frac{K_{11}}{M_{11}}} \tag{91}$$

or $K_{11} = (\omega_1)^2 M_{11}$. (92)

Likewise,

$$\begin{aligned} K_{22} &= (\omega_2)^2 M_{22} \, , \\ K_{33} &= (\omega_3)^2 M_{33} \end{aligned} \tag{93}$$

and so forth for all the degrees of freedom. It is now further assumed, and this must be carefully noted, that the degrees of freedom will be so chosen that there will be no elastic coupling, that is, so that

$$K_{jK} = 0 \quad \text{when } j \neq K. \tag{94}$$

(C ouring the degrees of freedom in this fashion is a common practice in the analysis of flutter stability.) A general expression for the k_{jK} is

$$k_{jK} = (\omega_L)^2 M_{LL} \delta_j^L \delta_K^L \quad , \tag{95}$$

whence substitution into (90) yields

$$Q_j = (\omega_j)^2 M_{jj} q^j . \tag{96}$$

These equations (91) through (96) are based on the mathematical relations expressing the vibratory motion of the system in only one of the given degrees of freedom at a time. A further pursuit of this line of thought, linked with the association of a coefficient of "structural" damping g'_j with each degree of freedom leads to a simple formula, analogous to (96), for P_j. This is

$$P_j = g'_j \, \omega_j \, M_{jj} \dot{q}^j . \tag{97}$$

The determination of g'_j from the logarithmic decrement δ'_j is simple, as follows

$$g'_j = \frac{2\delta'_j}{\sqrt{4\pi^2 + (\delta'_j)^2}} \tag{98}$$

$$\cong \delta'_j / \pi \qquad \text{when } \delta'_j \text{ is small.}$$

5. WORK AND ENERGY RELATIONS

While it adds nothing to the present formulation of the equations of motion, it is a valuable check on the basic formulation to investigate the work and energy relations. Using (15), (27), (28), (46), and (64) gives us

$$
\begin{aligned}
\frac{dT}{dt} &= \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik}\,\bar{v}_{ik}\cdot\frac{d\bar{v}_{ik}}{dt} \\
&= \sum_{i=1}^{N}\sum_{k=1}^{P_i} m_{ik}\,\frac{d\bar{v}_{ik}}{dt}\cdot\left(\bar{V}+\bar{\Omega}\times\bar{y}_{ik}+\frac{\partial\bar{y}_{ik}}{\partial q^{0}}+\frac{\partial\bar{y}_{ik}}{\partial q^{j}}\dot{q}^{j}\right) \\
&= \bar{V}\cdot\bar{F}+\bar{\Omega}\cdot\bar{G}+N_0+N_j\dot{q}^{j}\,. \qquad (99)
\end{aligned}
$$

$\frac{dU}{dt}$ and $\frac{dV}{dt}$ are given by (84) and (85), and, if W is work done by the external forces,

$$
\begin{aligned}
\frac{dW}{dt} &= \sum_{i=1}^{N}\sum_{k=1}^{P_i} \bar{F}_{ik}\cdot\bar{v}_{ik} \\
&= \sum_{i=1}^{N}\sum_{k=1}^{P_i} \bar{F}_{ik}\cdot\left(\bar{V}+\bar{\Omega}\times\bar{y}_{ik}+\frac{\partial\bar{y}_{ik}}{\partial q^{0}}+\frac{\partial\bar{y}_{ik}}{\partial q^{j}}\dot{q}^{j}\right) \\
&= \bar{V}\cdot\bar{F}+\bar{\Omega}\cdot\bar{G}+Q_0+Q_j\dot{q}^{j}, \qquad (100)
\end{aligned}
$$

use having been made of (47).

By substituting the suffix 0 for j in Equations (45) thru (49), we find that

$$N_0+O_0+P_0 = Q_0\,. \qquad (101)$$

Because of this and (49)

$$N_0+O_0+P_0+(N_j+O_j+P_j)\dot{q}^{j} = Q_0+Q_j\dot{q}^{j}, \qquad (102)$$

whence

$$\bar{V}\cdot\bar{F} + \bar{\Omega}\cdot\bar{G} + N_0 + O_0 + P_0 + (N_j + O_j + P_j)\dot{q}^j$$

$$= \bar{V}\cdot\bar{F} + \bar{\Omega}\cdot\bar{G} + Q_0 + Q_j q^j \ . \qquad (103)$$

[illegible] into (103) from ([illegible]), (84), (85), and (100) results in

$$\frac{dT}{dt} + \frac{dU}{dt} + \frac{dV}{dt} = \frac{dW}{dt} \ , \qquad (104)$$

or

$$T + U + V = W + \text{CONSTANT} , \qquad (105)$$

which simply states the fact that the work done by the external forces must be either stored in the form of kinetic or potential energy or dissipated.

6. PRACTICAL EXPRESSION OF THE INERTIAL FORMULAS

Not all of the foregoing equations are needed in the computational phase of investigating the elastic deformations of a vehicle in flight, but those that are essential for this purpose (in Sections 3 and 4), so far having been only rather abstractly expressed, need to be presented in terms that are suitable for practical use. Among these essential equations are (53), (56), (58), (59), and (60), and they are full of partial derivatives of $\bar{y}_{ih}$ (or its components) with respect to the generalized coordinates. For the purpose of computation, these partial derivatives need to be expressed in detail. For convenience, the subscripts i and h are temporarily dropped, and the basic notions of equations (1) thru (10) are developed and extended.

Let us introduce the vectors $\bar{h}_o$, $\bar{h}_K$, $\bar{\alpha}_o$, and $\bar{\alpha}_K$ having to do with the linear and angular velocities of the $\sim'$ coordinate system relative to the y system and defined as follows:

$$\bar{h}_o = \frac{\partial \bar{c}}{\partial q^o} = \bar{J}_r \frac{\partial c^r}{\partial q^o} = \bar{J}_r h_o^r \tag{106}$$

$$\bar{h}_K = \frac{\partial \bar{c}}{\partial q^K} = \bar{J}_r \frac{\partial c^r}{\partial q^K} = \bar{J}_r h_K^r \tag{107}$$

$$\bar{\alpha}_o = \bar{J}_1'\left(\bar{J}_3' \cdot \frac{\partial \bar{J}_2'}{\partial q^o}\right) + \bar{J}_2'\left(\bar{J}_1' \cdot \frac{\partial \bar{J}_3'}{\partial q^o}\right) + \bar{J}_3'\left(\bar{J}_2' \cdot \frac{\partial \bar{J}_1'}{\partial q^o}\right) \tag{108}$$

$$\bar{\alpha}_K = \bar{J}_1'\left(\bar{J}_3' \cdot \frac{\partial \bar{J}_2'}{\partial q^K}\right) + \bar{J}_2'\left(\bar{J}_1' \cdot \frac{\partial \bar{J}_3'}{\partial q^K}\right) + \bar{J}_3'\left(\bar{J}_2' \cdot \frac{\partial \bar{J}_1'}{\partial q^K}\right) \tag{109}$$

$\bar{h}_o$ and $\bar{h}_K$ are the partial linear velocities with respect to q^o and q^K of the origin of the $\bar{J}_r'$ coordinate system relative to the $\bar{J}_r$ system. $\bar{\alpha}_o$ and $\bar{\alpha}_K$ are the partial angular velocities with respect to q^o and q^K of the $\bar{J}_r'$ coordinate system relative to the $\bar{J}_r$ system.

There should be no particular difficulty in regard to the linear velocities, but some discussion of the angular velocities is definitely needed. First, let us note that

$$\bar{J}_r' \cdot \bar{J}_s' = \delta_{rs} = 1 \quad \text{when } r = s$$

$$= 0 \quad \text{when } r \neq s$$

Then

$$\frac{\partial}{\partial q^o}\left(\bar{J}_r' \cdot \bar{J}_s'\right) = \bar{J}_r' \cdot \frac{\partial \bar{J}_s'}{\partial q^o} + \bar{J}_s' \cdot \frac{\partial \bar{J}_r'}{\partial q^o} = 0\,. \tag{110}$$

Also, let us denote the components in the $\bar{J}'_r$ system of $\bar{\alpha}_0$ and $\bar{\alpha}_K$ respectively as α'^r_0 and α'^r_K. Then, from (108) and (110),

$$\left.\begin{aligned} \alpha'^1_0 &= \bar{J}'_3 \cdot \frac{\partial \bar{J}'_2}{\partial q^0} = -\bar{J}'_2 \cdot \frac{\partial \bar{J}'_3}{\partial q^0} \\ \alpha'^2_0 &= \bar{J}'_1 \cdot \frac{\partial \bar{J}'_3}{\partial q^0} = -\bar{J}'_3 \cdot \frac{\partial \bar{J}'_1}{\partial q^0} \\ \alpha'^3_0 &= \bar{J}'_2 \cdot \frac{\partial \bar{J}'_1}{\partial q^0} = -\bar{J}'_1 \cdot \frac{\partial \bar{J}'_2}{\partial q^0} \ ; \end{aligned}\right\} \tag{111}$$

and likewise, from (109),

$$\left.\begin{aligned} \alpha'^1_K &= \bar{J}'_3 \cdot \frac{\partial \bar{J}'_2}{\partial q^K} = -\bar{J}'_2 \cdot \frac{\partial \bar{J}'_3}{\partial q^K} \\ \alpha'^2_K &= \bar{J}'_1 \cdot \frac{\partial \bar{J}'_3}{\partial q^K} = -\bar{J}'_3 \cdot \frac{\partial \bar{J}'_1}{\partial q^K} \\ \alpha'^3_K &= \bar{J}'_2 \cdot \frac{\partial \bar{J}'_1}{\partial q^K} = -\bar{J}'_1 \cdot \frac{\partial \bar{J}'_2}{\partial q^K} \end{aligned}\right\} \tag{112}$$

These equations enable us to write

$$\bar{\alpha}_0 = \bar{J}'_r \alpha'^r_0 \quad \text{and} \quad \bar{\alpha}_K = \bar{J}'_r \ \alpha'^r_K \tag{113}$$

and to show that

$$\bar{\alpha}_0 \times \bar{J}'_r = \frac{\partial \bar{J}'_r}{\partial q^0} \quad \text{and} \quad \bar{\alpha}_K \times \bar{J}'_r = \frac{\partial \bar{J}'_r}{\partial q^K} \tag{114}$$

In further anticipation of terms to arise in the equations of motion, we derive from (111), (112), (113), and (114) the following relations:

$$\begin{aligned} \frac{\partial \alpha'^1_0}{\partial q^K} - \frac{\partial \alpha'^1_K}{\partial q^0} &= \frac{\partial \bar{J}'_3}{\partial q^K} \cdot \frac{\partial \bar{J}'_2}{\partial q^0} - \frac{\partial \bar{J}'_3}{\partial q^0} \cdot \frac{\partial \bar{J}'_2}{\partial q^K} \\ &= \alpha'^2_0 \alpha'^3_K - \alpha'^3_0 \alpha'^2_K \ . \end{aligned} \tag{115}$$

By symmetry, this can be extended and generalized to

$$\bar{J}'_r \left(\frac{\partial \alpha'^r_0}{\partial q^K} - \frac{\partial \alpha'^r_K}{\partial q^0} \right) = \bar{\alpha}_0 \times \bar{\alpha}_K \ . \tag{116}$$

From (113) and (114), it is readily found that

$$\frac{\partial\bar{\alpha}_o}{\partial q^{\kappa}} = \bar{J}'_r \frac{\partial\alpha'^r_o}{\partial q^{\kappa}} + \frac{\partial\bar{J}'_r}{\partial q^{\kappa}}\alpha'^r_o$$

$$= \bar{J}'_r \frac{\partial\alpha'^r_o}{\partial q^{\kappa}} + \bar{\alpha}_{\kappa} \times \bar{\alpha}_o . \tag{117}$$

Elimination of $\bar{\alpha}_o \times \bar{\alpha}_{\kappa}$ between (116) and (117) results in

$$\frac{\partial\bar{\alpha}_o}{\partial q^{\kappa}} = \bar{J}'_r \frac{\partial\alpha'^r_{\kappa}}{\partial q^o} \tag{118}$$

Likewise, $$\frac{\partial\bar{\alpha}_{\kappa}}{\partial q^o} = \bar{J}'_r \frac{\partial\alpha'^r_o}{\partial q^{\kappa}} , \tag{119}$$

and substitution from (118) and (119) into (116) results in

$$\frac{\partial\bar{\alpha}_{\kappa}}{\partial q^o} - \frac{\partial\bar{\alpha}_o}{\partial q^{\kappa}} = \bar{\alpha}_o \times \bar{\alpha}_{\kappa} . \tag{120}$$

The derivation of (115) thru (120) was of such generality that, in any of them, the suffix zero could be replaced by a letter. In the physical realm, this means that the relations expressed by these equations are applicable between degrees of freedom involving only small motions (type (3)) as well as between the large motions of type (2) and the motions of type (3).

In accordance with (8), $\frac{\partial v'^r}{\partial q^o} = 0$, but $\frac{\partial v'^r}{\partial q^{\kappa}}$ may or may not be zero. Let us introduce

$$\sigma'^r_{\kappa} = \frac{\partial v'^r}{\partial q^{\kappa}} \tag{121}$$

Then, since

$$\bar{v} = \bar{J}'_r v'^r ,$$

$$\frac{\partial\bar{v}}{\partial q^o} = \frac{\partial\bar{J}'_r}{\partial q^o} v'^r = \bar{\alpha}_o \times \bar{J}'_r v'^r$$

$$= \bar{\alpha}_o \times \bar{v} , \tag{122}$$

and $$\frac{\partial\bar{v}}{\partial q^{\kappa}} = \bar{J}'_r \frac{\partial v'^r}{\partial q^{\kappa}} + \frac{\partial\bar{J}'_r}{\partial q^{\kappa}} v'^r$$

$$= \bar{J}'_r \sigma'^r_{\kappa} + \bar{\alpha}_{\kappa} \times \bar{J}'_r v'^r$$

$$= \bar{\sigma}_{\kappa} + \bar{\alpha}_{\kappa} \times \bar{v}. \tag{123}$$

(1..), (1..), (122), and (123) facilitate the writing of the following

$$\frac{\partial \bar{y}}{\partial q^o} = \frac{\partial \bar{\sigma}}{\partial q^o} + \frac{\partial \bar{v}}{\partial q^o}$$

$$= \bar{h}_o + \bar{\alpha}_o \times \bar{v}, \qquad (124)$$

$$\frac{\partial \bar{y}}{\partial q^K} = \frac{\partial \bar{\omega}}{\partial q^K} + \frac{\partial \bar{v}}{\partial q^K}$$

$$= \bar{h}_K + \bar{\sigma}_K + \bar{\alpha}_K \times \bar{v} \qquad (125)$$

Let us introduce the two assumptions

$$\frac{\partial h_K^r}{\partial q^L} = 0 \qquad \text{and} \qquad \frac{\partial \sigma_K^r}{\partial q^L} = 0 \qquad (126)$$

and find expressions for the second partial derivatives of $\bar{y}$ with respect to the generalized coordinates. It can be found without too much trouble that, under these assumptions,

$$\frac{\partial^2 \bar{y}}{\partial q^o \partial q^o} = \frac{\partial \bar{h}_o}{\partial q^o} + \bar{\alpha}_o \times (\bar{\alpha}_o \times \bar{v}) + \frac{\partial \bar{\alpha}_o}{\partial q^o} \times \bar{v}, \qquad (127)$$

$$\frac{\partial^2 \bar{y}}{\partial q^o \partial q^K} = \bar{\alpha}_o \times (\bar{\sigma}_K + \bar{\alpha}_K \times \bar{v}) + \frac{\partial \bar{h}_o}{\partial q^K} + \frac{\partial \bar{\alpha}_o}{\partial q^K} \times \bar{v}, \qquad (128)$$

$$\frac{\partial^2 \bar{y}}{\partial q^K \partial q^L} = \bar{\alpha}_K \times \bar{\sigma}_L + \bar{\alpha}_L \times \bar{\sigma}_K + \bar{\alpha}_K \times (\bar{\alpha}_L \times \bar{v}) + \frac{\partial \bar{\alpha}_K}{\partial q^L} \times \bar{v}. \qquad (129)$$

The first important question that rises now is, how do we equate ( ) with (125)? In answer, let us observe that (125) is the general form of expression for $\partial \bar{y}/\partial q^K$; that the $\bar{a}_{Kih}$ are given as possible or tentative values of $\partial \bar{y}_{ih}/\partial q^K$, and that, therefore, the $\bar{a}_{Kih}$ will be given in the same form as (125). Thus, let the given linear velocity of the center of mass of section i be $\bar{\mathcal{J}}_{Ki}$, and let its given angular velocity be $\bar{\beta}_{Ki}$. Then

$$\bar{a}_{Kih} = \bar{\mathcal{J}}_{Ki} + \bar{\sigma}_{Kih} + \bar{\beta}_{Ki} \times \bar{v}_{ih}; \qquad (130)$$

and substitution from this into (40) results in

$$\frac{\partial \bar{y}_{ih}}{\partial q^K} = \bar{\mathcal{J}}_{Ki} + \bar{\sigma}_{Kih} + \bar{\beta}_{Ki} \times \bar{v}_{ih} + \bar{b}_K \times \bar{y}_{ih} + \bar{c}_K$$

$$= \bar{\mathcal{J}}_{Ki} + \bar{c}_K + \bar{b}_K \times (\bar{\theta}_i + \bar{v}_{ih}) + \bar{\sigma}_{Kih} + \bar{\beta}_{Ki} \times \bar{v}_{ih}$$

$$= \bar{h}_{Ki} + \bar{\sigma}_{Kih} + \bar{\alpha}_{Ki} \times \bar{v}_{ih} \qquad (131)$$

where $\quad \bar{h}_{\kappa i} = \mathcal{F}_{\kappa i} + \bar{\zeta}_{\kappa} + \bar{b}_{\kappa} \times \theta_i \qquad (132)$

and $\quad \bar{\sigma}_{\kappa i} = \bar{\mathcal{B}}_{\kappa i} + \bar{b}_{\kappa} \qquad (133)$

Let us now consider the practical problem of solving (43) for the $\bar{\bar{\omega}}$. With the use of $\bar{\theta}_i + \bar{v}_{ih}$ for $\bar{y}_{ih}$ and of Equation (130), the first term of (43) becomes

$$\sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih}\, \bar{y}_{ih} \times \bar{a}_{\kappa ih}$$

$$= \sum_{i=1}^{N} \sum_{h=1}^{P_i} m_{ih} (\bar{\theta}_i + \bar{v}_{ih}) \times (\mathcal{F}_{\kappa i} + \bar{\sigma}_{\kappa ih} + \bar{\mathcal{B}}_{\kappa i} \times \bar{v}_{ih})$$

$$= \sum_{i=1}^{N} \Big[m_i \bar{\theta}_i \times \mathcal{F}_{\kappa i} + \bar{\theta}_i \times \sum_{h=1}^{P_i} m_{ih}\, \bar{\sigma}_{\kappa ih}$$

$$+ \bar{\theta}_i \times (\bar{\mathcal{B}}_{\kappa i} \times \sum_{h=1}^{P_i} m_{ih} \bar{v}_{ih}) - \mathcal{F}_{\kappa i} \times \sum_{h=1}^{P_i} m_{ih} \bar{v}_{ih}$$

$$+ \sum_{h=1}^{P_i} m_{ih}\, \bar{v}_{ih} \times (\bar{\sigma}_{\kappa ih} + \bar{\mathcal{B}}_{\kappa i} \times \bar{v}_{ih}) \Big] . \qquad (134)$$

From (2), (24), and (121), the following results are obtained

$$\sum_{h=1}^{P_i} m_{ih} \bar{v}_{ih} = \bar{J}'_{ri} \sum_{h=1}^{P_i} m_{ih} v'^{r}_{ih} = 0 \quad , \qquad (135)$$

$$\sum_{h=1}^{P_i} m_{ih} \bar{\sigma}_{\kappa ih} = \bar{J}'_{ri} \sum_{h=1}^{P_i} m_{ih}\, \sigma'^{r}_{\kappa ih}$$

$$= \bar{J}'_{ri} \frac{\partial}{\partial q^{\kappa}} \sum_{h=1}^{P_i} m_{ih}\, v'^{r}_{ih} = 0 . \qquad (136)$$

This eliminates three terms of (134), and its final term is transformed as follows

$$\sum_{h=1}^{P_i} m_{ih}\, \bar{v}_{ih} \times (\bar{\mathcal{B}}_{\kappa i} \times \bar{v}_{ih})$$

$$= \sum_{h=1}^{P_i} m_{ih} [\bar{\mathcal{B}}_{\kappa i} (\bar{v}_{ih} \cdot \bar{v}_{ih}) - \bar{v}_{ih} (\bar{\mathcal{B}}_{\kappa i} \cdot \bar{v}_{ih})]$$

$$= \bar{J}'_{ri}\, \beta'^{s}_{Ki} \sum_{h=1}^{p_i} m_{ih} \left(\delta_{rs}\, v'^{t}_{ih}\, v'^{t}_{ih} - v'^{r}_{ih}\, v'^{s}_{ih} \right)$$

$$= \bar{J}'_{ri}\, \beta'^{s}_{Ki}\, H'_{rsi} , \tag{137}$$

where $$H'_{rsi} = \sum_{h=1}^{p_i} m_{ih} \left(\delta_{rs}\, v'^{t}_{ih}\, v'^{t}_{ih} - v'^{r}_{ih}\, v'^{s}_{ih} \right) . \tag{138}$$

The H'_{rsi} are easily seen to be the moments and the negatives of the products of inertia of section i about its own axes.

For convenience in treating the remaining terms of (134), we introduce the permutation symbol C_{rst} = 0 if two suffixes are the same
= 1 if (rst) is an even permutation of (123)
= -1 if (rst) is an odd permutation of (123),
the even permutations of (123) being (123), (231), and (312), and the odd permutations being (321), (213), and (132). Thus

$$\bar{J}_r \times \bar{J}_s = C_{rst}\, \bar{J}_t , \tag{139}$$

$$\sum_{i=1}^{N} m_i \bar{\theta}_i \times \bar{\mathcal{J}}_{Ki} = C_{rst}\, \bar{J}_t \sum_{i=1}^{N} m_i\, \theta^r_i\, \mathcal{J}^s_{Ki} , \tag{140}$$

and $$\sum_{h=1}^{p_i} m_{ih}\, \bar{v}_{ih} \times \bar{\sigma}_{Kih} = C_{rst}\, \bar{J}'_{ti}\, \Lambda'^{rs}_{Ki} , \tag{141}$$

where $$\Lambda'^{rs}_{Ki} = \sum_{h=1}^{p_i} m_{ih}\, v'^{r}_{ih}\, \sigma'^{s}_{Kih} . \tag{142}$$

Substitution from (135) thru (141) into (134) results in

$$\sum_{i=1}^{N} \sum_{h=1}^{p_i} m_{ih}\, \bar{y}_{ih} \times \bar{a}_{Kih} = \sum_{i=1}^{N} \Big(C_{rst}\, \bar{J}_t\, m_i\, \theta^r_i\, \mathcal{J}^s_{Ki} + C_{rst}\, \bar{J}'_{ti}\, \Lambda'^{rs}_{Ki} + \bar{J}'_{ri}\, \beta'^{s}_{Ki}\, H'_{rsi} \Big)$$

$$= \bar{J}_r \sum_{i=1}^{N} \Big(C_{rst}\, m_i\, \theta^s_i\, \mathcal{J}^t_{Ki} + C_{stu}\, e^r_{si}\, \Lambda'^{tu}_{Ki} + e^r_{si}\, H'_{sti}\, \beta'^{t}_{Ki} \Big) . \tag{143}$$

The second term of (43) is

$$\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\bar{y}_{ik}\times(\bar{b}_K\times\bar{y}_{ik})$$

$$=\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}[\bar{b}_K(\bar{y}_{ik}\cdot\bar{y}_{ik})-\bar{y}_{ik}(\bar{b}_K\cdot\bar{y}_{ik})]$$

$$=\bar{J}_r b_K^s\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}(\delta_{rs}y^t_{ik}y^t_{ik}-y^r_{ik}y^s_{ik})$$

$$=\bar{J}_r b_K^s I_{rs}\ , \qquad (144)$$

where

$$I_{rs}=\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}(\delta_{rs}y^t_{ik}-y^r_{ik}y^s_{ik})$$

$$=\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}[\delta_{rs}(\theta^t_i+e^t_{ui}v'^u_{ik})(\theta^t_i+e^t_{vi}v'^v_{ik})-(\theta^r_i+e^r_{ui}v'^u_{ik})(\theta^s_i+e^s_{vi}v'^v_{ik})]$$

$$=\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}[\delta_{rs}(\theta^t_i\theta^t_i+v'^t_{ik}v'^t_{ik})-(\theta^r_i\theta^s_i+e^r_{ui}e^s_{vi}v'^u_{ik}v'^v_{ik})]$$

$$=\sum_{i=1}^{N}[m_i(\delta_{rs}\theta^t_i\theta^t_i-\theta^r_i\theta^s_i)+e^r_{ui}e^s_{vi}\sum_{k=1}^{p_i}m_{ik}(\delta_{uv}v'^t_{ik}v'^t_{ik}-v'^u_{ik}v'^v_{ik})]$$

$$=\sum_{i=1}^{N}[m_i(\delta_{rs}\theta^t_i\theta^t_i-\theta^r_i\theta^s_i)+e^r_{ui}e^s_{vi}H'_{uvi}]\ . \qquad (145)$$

The I_{rs} are the moments and the negatives of the products of inertia of the vehicle about its axes.

The third term of (43) contains

$$\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\bar{a}_{Kik}=\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}(\bar{f}_{Ki}+\bar{\sigma}_{Kik}+\bar{B}_{Ki}\times\bar{v}_{ik})$$

$$=\sum_{i=1}^{N} m_i\bar{f}_{Ki}$$

$$=\bar{J}_r\sum_{i=1}^{N} m_i f^r_{Ki}\ , \qquad (146)$$

[illegible] is

$$m\bar{y}_c \times (\bar{\ell}_K \times \bar{y}_c) = m[\bar{\ell}_K(\bar{y}_c \cdot \bar{y}_c) - \bar{y}_c(\bar{\ell}_K \cdot \bar{y}_c)]$$

$$= m\bar{J}_r \ell_K^s (\delta_{rs} y_c^t y_c^t - y_c^r y_c^s) \quad . \tag{147}$$

Substitution from (143) thru (147) into (43) results in

$$\bar{J}_r \sum_{i=1}^{N} (C_{rst} m_i \theta_i^s \mathcal{J}_{Ki}^t + C_{stu} e_{si}^r \Lambda_{Ki}^{-tu} + e_{si}^r H_{sti}' \mathcal{B}_{Ki}^{-t})$$
$$+ \bar{J}_r \ell_{K}^s I_{rs} - \bar{y}_c \times \bar{J}_t \sum_{i=1}^{N} m_i \mathcal{J}_{Ki}^t$$
$$- \bar{J}_r \ell_K^s m(\delta_{rs} y_c^t y_c^t - y_c^r y_c^s) = 0 \quad . \tag{148}$$

Noting that the form $\bar{y}_c \times \bar{J}_t = C_{rst} \bar{J}_r y_c^s$, we can put this in

$$[I_{rs} - m(\delta_{rs} y_c^t y_c^t - y_c^r y_c^s)] \ell_K^s = C_{rst} y_c^s \sum_{i=1}^{N} m_i \mathcal{J}_{Ki}^t$$
$$- \sum_{i=1}^{N} (C_{rst} m_i \theta_i^s \mathcal{J}_{Ki}^t + C_{stu} e_{si}^r \Lambda_{Ki}^{-tu} + e_{si}^r H_{sti}' \mathcal{B}_{Ki}^{-t}) \quad . \tag{149}$$

Let $H_K^t = \sum_{i=1}^{N} m_i \mathcal{J}_{Ki}^t$ (150)

and $L_K^{st} = \sum_{i=1}^{N} m_i \theta_i^s \mathcal{J}_{Ki}^t$; (151)

then (149) becomes

$$[I_{rs} - m(\delta_{rs} y_c^t y_c^t - y_c^r y_c^s)] \ell_K^s = C_{rst} y_c^s H_K^t$$
$$- C_{rst} L_K^{st} - \sum_{i=1}^{N} (C_{stu} e_{si}^r \Lambda_{Ki}^{-tu} + e_{si}^r H_{sti}' \mathcal{B}_{Ki}^{-t}) \, , \tag{152}$$

which can be solved for the ℓ_K^s by familiar techniques. With the aid of (41), (146), (150), and the fact that $\bar{\ell}_K \times \bar{y}_c = \bar{J}_r C_{rst} \ell_K^s y_c^t$, it is easily seen that

$$C_K^r = C_{rst} y_c^s \ell_K^t - \frac{1}{m} H_K^r \quad . \tag{153}$$

From ( ),

$$\bar{J}_r \alpha^r_{Kih} = \bar{J}_r \gamma^r_{Ki} + \bar{J}'_{si} \sigma'^s_{Kih} + \bar{J}'_{ti} \times \bar{J}'_{ui} B'^t_{Ki} \nu'^u_{ih}$$

$$= \bar{J}_r \left(\gamma^r_{Ki} + e^r_{si} \sigma'^s_{Kih} \right) + \bar{J}'_{si} C_{stu} B'^t_{Ki} \nu'^u_{ih}$$

$$= \bar{J}_r \left[\gamma^r_{Ki} + e^r_{si} \left(\sigma'^s_{Kih} + C_{stu} B'^t_{Ki} \nu'^u_{ih} \right) \right],$$

or

$$\alpha^r_{Kih} = \gamma^r_{Ki} + e^r_{si} \left(\sigma'^s_{Kih} + C_{stu} B'^t_{Ki} \nu'^u_{ih} \right). \quad (154)$$

Similarly (40) is transformed to

$$\bar{J}_r \frac{\partial y^r_{ih}}{\partial q^K} = \bar{J}_r \alpha^r_{Kih} + \bar{J}_s \times \bar{J}_t b^s_K y^t_{ih} + \bar{J}_r C^r_K$$

$$= \bar{J}_r \left(\alpha^r_{Kih} + C_{rst} b^s_K y^t_{ih} + C^r_K \right)$$

or

$$\frac{\partial y^r_{ih}}{\partial q^K} = C^r_K + C_{rst} b^s_K y^t_{ih} + \gamma^r_{Ki}$$

$$+ e^r_{si} \left(\sigma'^s_{Kih} + C_{stu} B'^t_{Ki} \nu'^u_{ih} \right). \quad (155)$$

We now turn our attention to the evaluation of the β_{rsj}, defined in (53).

$$\sum_{i=1}^{N} \sum_{h=1}^{p_i} m_{ih} y^r_{ih} \frac{\partial y^s_{ih}}{\partial q^j} = \sum_{i=1}^{N} \sum_{h=1}^{p_i} m_{ih} y^r_{ih} \left[C^s_j + C_{stu} b^t_j y^u_{ih} \right.$$

$$\left. + \gamma^s_{ji} + e^s_{ti} \left(\sigma'^t_{jih} + C_{tuv} B'^u_{ji} \nu'^v_{ih} \right) \right]. \quad (156)$$

[illegible] terms of ([illegible]), [illegible] find

$$\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\, y^r_{ik}\, c^s_j = m\, y^r_c\, c^s_j\,, \tag{157}$$

$$\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\, y^r_{ik}\, C_{stu}\, b^t_j\, y^u_{ik} = C_{stu}\, b^t_j\, G_{ru}\,, \tag{158}$$

where $G_{ru} = \sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\, y^r_{ik}\, y^u_{ik}$

$$= \delta_{ru} I_{vv}/2 - I_{ru}\,; \tag{159}$$

$$\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\, y^r_{ik}\, \gamma^s_{ji} = \sum_{i=1}^{N} m_i\, \theta^r_i\, \gamma^s_{ji}\,; \tag{160}$$

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\, y^r_{ik}\, e^s_{ti}\, \sigma^t_{jik} \\
&\quad = \sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\left(\theta^r_i + e^r_{ui}\, v^u_{ik}\right) e^s_{ti}\, \sigma^t_{jik} \\
&\quad = \sum_{i=1}^{N} e^r_{ui}\, e^s_{ti} \sum_{k=1}^{p_i} m_{ik}\, v^u_{ik}\, \sigma^t_{jik} \\
&\quad = \sum_{i=1}^{N} e^r_{ui}\, e^s_{ti}\, \Lambda^{ut}_{ji}\,;
\end{aligned} \tag{161}$$

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{p_i} m_{ik}\, y^r_{ik}\, e^s_{ti}\, C_{tuv}\, B^u_{ji}\, v^v_{ik} \\
&\quad = C_{tuv} \sum_{i=1}^{N} e^s_{ti}\, B^u_{ji} \sum_{k=1}^{p_i} m_{ik}\left(\theta^r_i + e^r_{pi}\, v^p_{ik}\right) v^v_{ik} \\
&\quad = C_{tuv} \sum_{i=1}^{N} e^s_{ti}\, B^u_{ji}\, e^r_{pi}\, \Gamma_{pvi}\,,
\end{aligned} \tag{162}$$

$$\ldots\; \Gamma'_{\rho\nu i} = \sum_{h=1}^{\rho} m_{ih}\, v'^{\rho}_{ih}\, v'^{\nu}_{ih}$$

$$- \delta_{\rho\nu} H'_{qqi}/2 - H'_{\rho\nu i} \qquad (163)$$

Substitution from (157) thru (162) into (156) results in

$$\sum_{i=1}^{\nu} \sum_{h=1}^{\rho} m_{ih} \cdot y^{r}_{ih} \frac{\partial y^{s}_{ih}}{\partial q^{j}} = m y_{c} C^{s}_{j} + C_{stu}\, b^{t}_{j}\, G_{ru} \qquad (164)$$

$$+ \sum_{i=1}^{\nu} [m_{i} o^{r}_{i} \gamma^{s}_{ji} + e^{r}_{ui}\, e^{s}_{ti}\, \Lambda'^{ut}_{ji} + C_{tuv}\, e^{s}_{ti}\, e^{r}_{\rho i}\, \beta'^{u}_{ji} \Gamma'_{\rho\nu i}].$$

From this, setting s equal to r (and summing over r),

$$\sum_{i=1}^{\nu} \sum_{h=1}^{\rho} m_{ih}\, y^{r}_{ih} \frac{\partial y^{r}_{ih}}{\partial q^{j}} = \sum_{i=1}^{\nu} \sum_{h=1}^{\rho} m_{ih}\, y^{r}_{ih} \frac{\partial y^{r}_{ih}}{\partial q^{j}}$$

$$= m y^{r}_{c} C^{r}_{j} + C_{rtu}\, b^{t}_{j}\, G_{ru}$$

$$+ \sum_{i=1}^{\nu} [m_{i} o^{r}_{i} \gamma^{r}_{ji} + e^{r}_{ui}\, e^{r}_{ti}\, \Lambda'^{ut}_{ji} + C_{tuv}\, e^{r}_{ti}\, e^{r}_{\rho i}\, \beta'^{u}_{ji} \Gamma'_{\rho\nu i}]$$

$$= m y^{r}_{c} C^{r}_{j} + \sum_{i=1}^{\nu} (m_{i} o^{r}_{i} \gamma^{r}_{ji} + \Lambda'^{tt}_{ji}), \qquad (165)$$

since $G_{ru} = G_{ur}$, $e^{r}_{ui} e^{r}_{ti} = \delta_{ut}$; and

$$\Gamma'_{\rho\nu i} = \Gamma'_{\nu\rho i}.$$

P_{rsj} is now obtained by substitution from (164) and (165) into (53).

$$P_{rsj} = \delta_{rs}[m y_c^t C_j^t + \sum_{i=1}^{N} m_i \sigma_i^t \gamma_{ji}^t + \Lambda_{ji}^{tt}] - m y_c^r C_j^s$$

$$- C_{..u} b_j^t G_{..} - \sum_{i=1}^{N} (m_i \sigma_i^r \gamma_{ji}^s + e_{ti}^r e_{ui}^s \Lambda_{ji}^{tu}$$

$$+ C_{tuv} e_{ti}^s e_{pi}^r \beta_{ji}^u \Gamma_{pvi})$$

$$= m(\delta_{rs} y_c^t C_j^t - y_c^r C_j^s) - C_{stu} b_j^t G_{ru}$$

$$+ \sum_{i=1}^{N} [m_i (\delta_{rs} \sigma_i^t \gamma_{ji}^t - \sigma_i^r \gamma_{ji}^s) + \delta_{rs} \Lambda_{ji}^{tt} - e_{ti}^r e_{ui}^s \Lambda_{ji}^{tu}$$

$$- C_{tuv} e_{pi}^r e_{ti}^s \beta_{ji}^u \Gamma_{pvi}]$$

$$= m(\delta_{rs} y_c^t C_j^t - y_c^r C_j^s) - C_{stu} b_j^t G_{ru}$$

$$+ \delta_{rs} L_j^{tt} - L_j^{rs} + \sum_{i=1}^{N} [e_{ti}^r e_{ui}^s (\delta_{tu} \Lambda_{ji}^{vv} - \Lambda_{ji}^{tu})$$

$$- C_{tuv} e_{pi}^r e_{ti}^s \beta_{ji}^u \Gamma_{pvi}] \tag{166}$$

Let $\mu_{jk} = \sum_{i=1}^{N} \sum_{k=1}^{B} m_{ik} \sigma_{jik}^{r} \sigma_{kik}^{r}$, (167)

and substitute from (155) into (60).

Then, after simplification and making use of (22), (23), (25), (135), (136), (138), (142), (145), (150), (151), and (159), the result is obtained that

$$M_{jk} = m[C_j^r C_k^r + C_{rst}(C_j^r b_k^s + C_k^r b_j^s) y_c^t]$$

$$+ \mu_{jk} + b_j^r b_k^s I_{rs} + C_j^r H_k^r + C_k^r H_j^r$$

$$+ C_{rst}(b_j^r L_k^{st} + b_k^r L_j^{st}) + \sum_{i=1}^{N} m_i \gamma_{ji}^r \gamma_{ki}^r$$

$$+ C_{rst}\left(b_j^u \sum_{i=1}^{N} e_{ri}^u \Lambda_{Ki}^{\prime st} + b_K^u \sum_{i=1}^{N} e_{ri}^u \Lambda_{ji}^{\prime st}\right)$$

$$+ C_{rst} \sum_{i=1}^{N} \left(\beta_{ji}^{\prime r} \Lambda_{Ki}^{\prime st} + \beta_{Ki}^{\prime r} \Lambda_{ji}^{\prime st}\right)$$

$$+ b_j^r \sum_{i=1}^{N} e_{si}^r \beta_{ri}^{\prime t} H_{sti}^{\prime} + b_K^r \sum_{i=1}^{N} e_{si}^r \beta_{Ki}^{\prime t} H_{sti}^{\prime}$$

$$+ \sum_{i=1}^{N} \beta_{ji}^{\prime r} \beta_{Ki}^{\prime s} H_{rsi}^{\prime} \tag{168}$$

This equation is confirmed by writing out an expression for the kinetic energy T and making use of (72).

To provide more compact notation, let

$$D_{jK} = \sum_{i=1}^{N} m_i \gamma_{ji}^r \gamma_{Ki}^r , \tag{169}$$

$$\Lambda_K^r = C_{stu} \sum_{i=1}^{N} e_{si}^r \Lambda_{Ki}^{\prime tu} , \tag{170}$$

$$\Delta_{jK} = C_{rst} \sum_{i=1}^{N} \left(\beta_{ji}^{\prime r} \Lambda_{Ki}^{\prime st} + \beta_{Ki}^{\prime r} \Lambda_{ji}^{\prime st}\right), \tag{171}$$

$$N_K^r = \sum_{i=1}^{N} e_{si}^r \beta_{Ki}^{\prime t} H_{sti}^{\prime} ; \quad \text{and} \tag{172}$$

$$H_{jK} = \sum_{i=1}^{N} \beta_{ji}^{\prime r} \beta_{Ki}^{\prime s} H_{rsi}^{\prime} \tag{173}$$

$$\text{Then} \quad M_{jK} = m[c_j^r c_K^r + C_{rst}\left(c_j^r b_K^s + c_K^r b_j^s\right) y_c^t]$$
$$+ b_j^r b_K^s I_{rs} + c_j^r H_K^r + c_K^r H_j^r$$
$$+ b_j^r \left(C_{rst} L_K^{st} + \Lambda_K^r + N_K^r\right)$$
$$+ b_K^r \left(C_{rst} L_j^{st} + \Lambda_j^r + N_j^r\right)$$
$$+ D_{jK} + \Delta_{jK} + H_{jK} + \mu_{jK} . \tag{174}$$

If the subscript i in (120) is replaced by L and the result applied to (129), it can be demonstrated that interchanging K and L does not change the value of (129), which is as it should be. This "symmetry" of (129) depends on the retention of the last term; but $\partial\bar{\alpha}_K/\partial q^L$, which is a factor in the last term, is difficult to obtain and likely to be small; therefore, a means of dropping it out without destroying the symmetry of the equation is sought. This is accomplished by a simple averaging, as follows:

$$\frac{\partial^2\bar{y}}{\partial q^K \partial q^L} = \frac{\partial^2\bar{y}}{\partial q^L \partial q^K} = \frac{1}{2}\left(\frac{\partial^2\bar{y}}{\partial q^K \partial q^L} + \frac{\partial^2\bar{y}}{\partial q^L \partial q^K}\right) \tag{175}$$

$$\cong \bar{\alpha}_K \times \bar{\sigma}_L + \bar{\alpha}_L \times \bar{\sigma}_K + \frac{1}{2}\left[\bar{\alpha}_K \times (\bar{\alpha}_L \times \bar{v}) + \bar{\alpha}_L \times (\bar{\alpha}_K \times \bar{v})\right]$$

$$= \bar{\alpha}_K \times \bar{\sigma}_L + \bar{\alpha}_L \times \bar{\sigma}_K - (\bar{\alpha}_K \cdot \bar{\alpha}_L)\bar{v} + \frac{1}{2}\left[(\bar{\alpha}_K \cdot \bar{v})\bar{\alpha}_L + (\bar{\alpha}_L \cdot \bar{v})\bar{\alpha}_K\right] .$$

Employing (125) and (175) in (59) results in

$$\boxed{KL,j} \cong \sum_{i=1}^{N} \sum_{k=1}^{p_i} m_{ik}\left(\bar{h}_{ji} + \bar{\sigma}_{jik} + \bar{\alpha}_{ji} \times \bar{v}_{ik}\right) \cdot \Big\{\bar{\alpha}_{Ki} \times \bar{\sigma}_{Lik} + \bar{\alpha}_{Li} \times \bar{\sigma}_{Kik} \tag{176}$$

$$- (\bar{\alpha}_{Ki} \cdot \bar{\alpha}_{Li})\bar{v}_{ik} + \frac{1}{2}\left[(\bar{\alpha}_{Ki} \cdot \bar{v}_{ik})\bar{\alpha}_{Li} + (\bar{\alpha}_{Li} \cdot \bar{v}_{ik})\bar{\alpha}_{Ki}\right]\Big\} .$$

If this is now expanded, the result is

$$\boxed{KL,j} \cong \sum_{i=1}^{N} \Big[c_{rst}\left(\alpha'^{r}_{Ki} S'^{st}_{Lji} + \alpha'^{r}_{Li} S'^{st}_{Kji}\right) + \alpha'^{r}_{Ki} \alpha'^{s}_{Li} Q'^{rs}_{ji}$$

$$+ \left(\alpha'^{r}_{Ki} P'^{rs}_{Li} + \alpha'^{r}_{Li} P'^{rs}_{Ki}\right) \alpha'^{s}_{ji}\Big] , \tag{177}$$

where

$$S'^{rs}_{jKi} = \sum_{k=1}^{p_i} m_{ik}\, \sigma'^{r}_{jik}\, \sigma'^{s}_{Kik} , \tag{178}$$

$$P'^{rs}_{ji} = \sum_{k=1}^{p_i} m_{ik}\left(\delta_{rs}\, v'^{t}_{ik}\, \sigma'^{t}_{jik} - v'^{r}_{ik}\, \sigma'^{s}_{jik}\right)$$

$$= \delta_{rs}\, \Lambda'^{tt}_{ji} - \Lambda'^{rs}_{ji} , \tag{179}$$

and $a_{ji}^{-rs} = \frac{1}{2}\left[\alpha_{ji}^{-t}\left(C_{stu}\Gamma_{rui}' + C_{rtu}\Gamma_{sui}'\right)\right.$

$$\left. - p_{ji}^{-rs} - p_{ji}^{-sr}\right]. \tag{180}$$

If we note that $S_{jKi}^{-rs} = 0$ when j and/or K equal zero and that $p_{ji}^{-rs} = 0$ when $j = 0$, then it follows from (177) that

$$\boxed{OK,j} \cong \sum_{i=1}^{N}\left(C_{rst}\alpha_{oi}^{-r}S_{Kji}^{-st} + \alpha_{oi}^{-r}p_{Ki}^{-rs}\alpha_{ji}^{-s}\right.$$

$$\left. + \alpha_{Ki}^{-r}\alpha_{oi}^{-s}a_{ji}^{-rs}\right) \tag{181}$$

and $$\boxed{Ooj} \cong \sum_{i=1}^{N}\alpha_{oi}^{-r}\alpha_{oi}^{-s}a_{ji}^{-rs} . \tag{182}$$

7. PRACTICAL EXPRESSION OF THE AERODYNAMIC FORMULAS

The aerodynamic forces, being external, are accounted for by the use of Equation (47). In this use of (47), however, only those particles lying on the surface of the vehicle will be involved. In the present formulation, we have recourse to the simplest available aerodynamic theory that offers sufficient generality, namely, Newtonian flow theory. Let $\overline{n}$ be a unit vector located at a certain point on the surface, perpendicular to the surface at that point, and pointing outward. The velocity of that point is $\overline{v}$, as given in (15). Letting ρ be the atmospheric density, then, according to Newtonian flow theory, the aerodynamic force per unit area ($\overline{F}$) at the given point on the surface of the vehicle is as follows:

1. When $\overline{n} \cdot \overline{v} \leq 0, \quad \overline{F} = 0.$ (183)
2. When $\overline{n} \cdot \overline{v} > 0, \quad \overline{F} = -\overline{n}\rho(\overline{n} \cdot \overline{v})^2.$ (184)

The scalar $\overline{n} \cdot \overline{v}$ may be called the "piston speed" of the given point (or the downwash at that point) and is symbolized by w; thus, when $w > 0$,

$$\overline{F} = -\overline{n}\rho w^2. \tag{185}$$

For the purpose of evaluating w, it is noted that the first and last terms of $\overline{v}$ as given in (15) are, under normal conditions, much more significant than the two middle terms. Dropping these less significant terms results in

$$w = \overline{n} \cdot \left(\overline{V} + \frac{\partial \overline{y}}{\partial q^K} \dot{q}^K\right). \tag{186}$$

A considerable practical advantage can be realized if w^2 in (185) is replaced by a linear approximation (or expansion) about the elastically undeformed configuration. Regarding w as a function of q^K and $\dot{q}^K$ for this purpose, and using the subscript o to denote the undeformed configuration, we obtain

$$w^2 = w_o^2 + 2w_o\left(\frac{\partial w_o}{\partial q_o^K} q^K + \frac{\partial w_o}{\partial \dot{q}_o^K} \dot{q}^K\right). \tag{187}$$

Now

$$\left.\begin{aligned} w_o &= \overline{V} \cdot \overline{n}, \\ \frac{\partial w_o}{\partial q_o^K} &= \overline{V} \cdot \frac{\partial \overline{n}}{\partial q_o^K}, \\ \frac{\partial w_o}{\partial \dot{q}_o^K} &= \overline{n} \cdot \frac{\partial \overline{y}}{\partial q_o^K}; \end{aligned}\right\} \tag{188}$$

and, therefore,

$$w^2 = (\overline{V} \cdot \overline{n})^2 + 2\overline{V} \cdot \overline{n}\left(\overline{V} \cdot \frac{\partial \overline{n}}{\partial q_o^K} q^K + \overline{n} \cdot \frac{\partial \overline{y}}{\partial q_o^K} \dot{q}^K\right). \tag{189}$$

In applying (47) to the calculation of the generalized air dynamic forces, we are impelled to replace the summation over k with an integration over the surface of the i-th section by virtue of the fact that only points on the surface are involved. Let S_i be the surface the i-th section then

$$
\begin{aligned}
Q_i &= -\sum_{i=1}^{N}\int_{S_i} \frac{\partial \bar{y}}{\partial q^j}\cdot \bar{F}\,ds \\
&= -\sum_{i=1}^{k}\int_{S_i} \frac{\partial \bar{y}}{\partial q^j}\cdot \bar{n}\,\rho\, w^2\, ds \\
&= -\rho\sum_{i=1}^{k}\int_{S_i}\xi_j\Big[(\bar{V}\cdot\bar{n})^2 \\
&\qquad + 2\bar{V}\cdot\bar{n}\Big(\bar{V}\,\frac{\partial \bar{n}}{\partial q_0^k}q^k + \xi_k\dot{q}^k\Big)\Big]ds\,;
\end{aligned}
\tag{190}
$$

where $$\xi_j = \bar{n}\cdot\frac{\partial \bar{y}}{\partial q_0^j}\;. \tag{191}$$

The following development of formulas serves to make this more practical for numerical computations

$$\bar{n} = \bar{J}'_r\, n'^r. \tag{192}$$

$$
\begin{aligned}
\frac{\partial \bar{n}}{\partial q^k} &= \frac{\partial \bar{J}'_r}{\partial q^k}\, n'^r + \bar{J}'_r\,\frac{\partial n'^r}{\partial q^k} \\
&= \bar{\alpha}_k \times \bar{n} + \bar{J}'_r\,\frac{\partial n'^r}{\partial q^k} \\
&= \bar{J}'_r\Big(c_{rst}\,\alpha'^s_k\, n'^t + \frac{\partial n'^r}{\partial q^k}\Big)\;.
\end{aligned}
\tag{193}
$$

Inserting this into [illegible] and transforming it in other ways leads to

$$Q_j = -\rho \sum_{i=1}^{N} \int_{s_i} \xi_j \Big[V^r V^s (\bar{J}_r \cdot \bar{J}'_{ti})(\bar{J}_s \cdot \bar{J}'_{ui})\, \eta^{-t} \eta^{-u}$$

$$+ 2 V^r V^s (\bar{J}_r \cdot \bar{J}'_{ti})(\bar{J}_s \cdot \bar{J}'_{ui})\, \eta^{-t} \Big(C_{uvx}\, \alpha^{-v}_{\ k}\, \eta^{-x} + \frac{\partial \eta^{-u}}{\partial q^k} \Big) q^k$$

$$+ 2 V^r (\bar{J}_r \cdot \bar{J}'_{ti})\, \eta^{-t} \xi_k \dot{q}^k \Big] ds$$

$$= -\rho V^r V^s \sum_{i=1}^{N} e^r_{ti} e^s_{ui} \int_{s_i} \xi_j \eta^{-t} \eta^{-u} ds$$

$$- 2\rho V^r V^s \sum_{i=1}^{N} e^r_{ti} e^s_{ui} \Big(C_{uvx}\, \alpha^{-v}_{ki} \int_{s_i} \eta^{-t} \eta^{-x} \xi_j \, ds$$

$$+ \int_{s_i} \xi_j \eta^{-t} \frac{\partial \eta^{-u}}{\partial q^k} ds \Big) q^k$$

$$- 2\rho V^r \sum_{i=1}^{N} e^r_{ti} \int_{s_i} \eta^{-t} \xi_j \xi_k \, ds \, \dot{q}^k$$

$$= -\rho \left(V^r V^s A^{rs}_j + 2 V^r V^s B^{rs}_{jk} q^k + 2 V^r C^r_{jk} \dot{q}^k \right), \qquad (194)$$

where
$$A^{rs}_j = \sum_{i=1}^{N} e^r_{ti} e^s_{ui} \int_{s_i} \xi_j \eta^{-t} \eta^{-u} ds, \qquad (195)$$

$$B^{rs}_{jk} = \sum_{i=1}^{N} e^r_{ti} e^s_{ui} \Big(C_{uvx}\, \alpha^{-v}_{ki} \int_{s_i} \xi_j \eta^{-t} \eta^{-x} ds$$

$$+ \int_{s_i} \xi_j \eta^{-t} \frac{\partial \eta^{-u}}{\partial q^k} ds \Big), \qquad (196)$$

and
$$C^r_{jk} = \sum_{i=1}^{N} e^r_{ti} \int_{s_i} \xi_j \xi_k \eta^{-t} ds. \qquad (197)$$

Substitution from (125) into (191) results in

$$\xi_j = \bar{\eta} \cdot (\bar{h}_j + \bar{\sigma}_j + \bar{\alpha}_j \times \bar{v})$$

$$= \eta^{-o} \left(e^{p}_{o} h^{p}_j + \sigma^{-o}_j + C_{opq}\, \alpha^{-p}_j v^{-q} \right). \qquad (198)$$

[illegible]

It is assumed (or introduced as a limitation on the applicability of this formulation) that all thrust forces are directed by thrust "vectoring" (that is, directing) nozzles and that all such nozzles are symmetric with respect to the axis (or line) of thrust. Regarding each thrust vectoring nozzle as a rigid but movable structural section identifiable by a subscript i, we place therein a triad of unit vectors $\bar{J}'_{ri}$ with $\bar{J}'_{1i}$ pointing in the direction of the thrust and coinciding with the line of thrust. Because of the symmetry of the nozzle, its center of mass will lie on the thrust axis and the usual condition that the origin of the $\bar{J}'_{ri}$ triad be at the center of mass of the section can be complied with.

At the center of mass of a section, $\frac{\partial \bar{r}_i}{\partial q^j} = \frac{\partial \bar{z}_i}{\partial q^j} = \bar{h}_{ji}$.
Representation of the thrust force at the i-th nozzle as $\bar{J}'_{1i} T_i$ and substitution into (47) results in the following expression for the generalized forces associated with the thrust forces:

$$
\begin{aligned}
Q_j &= \sum_{i=1}^{E} \bar{h}_{ji} \cdot \bar{J}'_{1i} T_i \\
&= \sum_{i=1}^{E} \bar{J}_{ri} \cdot \bar{J}'_{1i} h^r_{ji} T_i \\
&= \sum_{i=1}^{E} e^r_{1i} h^r_{ji} T_i ,
\end{aligned}
\tag{199}
$$

E being the number of thrust vectoring nozzles (or "engines").

As has already been indicated in (7), the e^r_{1i} are functions of q^ν and the q^k. The correct inclusion of the dependence of the e^r_{1i} on the q^k would be the best procedure and would enable the program to reveal the interaction between thrust and elastic deformation even to the point of detecting instabilities if any existed. However, doing this would impose considerable additional difficulty and go beyond the scope of the program; therefore, the dependence of the e^r_{1i} on the q^k will be disregarded here. On the other hand, their dependence on q^ν must be and is included as shown in the following section.

9. DIRECTION COSINES OF MOVABLE STRUCTURAL SECTION

In addition to thrust vectoring nozzles, there are such movable structural sections as control surfaces of various types. For the sake of simplicity it is assumed that the large motions of all control surfaces consist of nothing more than a rotation about a fixed axis. It is convenient to place the $\bar{J}'_{1i}$ vector of such a section parallel to, but not necessarily on, this axis of rotation. Doing this makes it possible to employ Eulerian angles to define the orientation of both thrust vectoring nozzles and control surfaces.

These angles are shown in Figure 3 and defined as follows.

ϕ_i = angle of rotation of the plane and axis of swivel about the y' axis ($\bar{J}_1$).

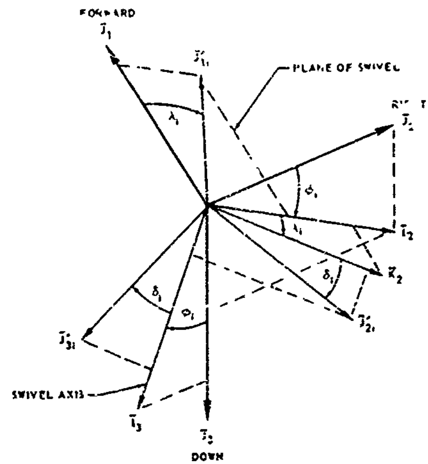
λ_i = angle of swivel of nozzle (or the $\bar{J}'_{1i}$ vector) about an axis ($\bar{\iota}_3$) perpendicular to $\bar{J}_1$ and making an angle ϕ_i with $\bar{J}_3$.

δ_i = angle of rotation of $\bar{J}'_{2i}$ and $\bar{J}'_{3i}$ about $\bar{J}'_{1i}$.

By familiar processes of vector analysis and classical mechanics, it is known that the direction cosines relating the $\bar{J}'_{ri}$ vectors to the $\bar{J}_s$ vectors are the following e^s_{ri}:

$$\left.\begin{aligned}
e^1_{1i} &= \cos\lambda_i \\
e^1_{2i} &= -\sin\lambda_i \cos\delta_i \\
e^1_{3i} &= \sin\lambda_i \sin\delta_i \\
e^2_{1i} &= \cos\phi_i \sin\lambda_i \\
e^2_{2i} &= \cos\phi_i \cos\lambda_i \cos\delta_i - \sin\phi_i \sin\delta_i \\
e^2_{3i} &= -\cos\phi_i \cos\lambda_i \sin\delta_i - \sin\phi_i \cos\delta_i \\
e^3_{1i} &= \sin\phi_i \sin\lambda_i \\
e^3_{2i} &= \sin\phi_i \cos\lambda_i \cos\delta_i + \cos\phi_i \sin\delta_i \\
e^3_{3i} &= -\sin\phi_i \cos\lambda_i \sin\delta_i + \cos\phi_i \cos\delta_i
\end{aligned}\right\} \qquad (200)$$

It is understood that ϕ_i, λ_i, and δ_i are functions of φ as determined by autopilot or flight programmer commands.



NOTE:

a) $\bar{I}_2$ AND $\bar{I}_3$ ARE ⊥ , ROTATE ABOUT AND ARE ⊥ TO $\bar{J}_1$

b) $\bar{J}'_{1i}$ AND $\bar{K}_2$ ARE ⊥ , [illegible] ABOUT AND ARE ⊥ TO $\bar{I}_3$

c) $\bar{J}'_{2i}$ AND $\bar{J}'_{3i}$ ARE ⊥ , ROTATE ABOUT AND ARE ⊥ TO $\bar{J}'_{1i}$

Figure 3. Orientation Angles of Movable Sections

1[illegible]. FORMULAS FOR THE STRUCTURAL LOADS

The shear force $\overline{S}$ at a specified location on the vehicle is the negative of the sum of the internal forces exerted by all the particles of the vehicle on the particles located on one side of the chosen shear plane. For the sake of economy, the side of the shear plane selected for this purpose will be the side on which the smaller number of particles is found. This will usually be the side away from (or lying outboard of) the center of mass. Let the absence of specific designation as to which particles and sections are included in a summation be understood to mean summation over the particles on the chosen side of the shear plane. Then, with the aid of (44),

$$\overline{S} = -\sum_i \sum_k \sum_{\ell=1}^{N} \sum_{j=1}^{P_i} \overline{F}_{ik\ell j} = \sum_i \sum_k \left(\overline{F}_{ik} - m_{ik} \frac{d\overline{v}_{ik}}{dt} \right)$$

$$= \sum_i \sum_k \overline{F}_{ik} - \sum_i \sum_k m_{ik} \frac{d\overline{v}_{ik}}{dt} \,. \qquad (201)$$

Except for the number of particles included in the summation, the last term of (201) is the same as the right side of (27). A practical symmetric expression for $\frac{\partial^2 \overline{y}}{\partial q^r \partial q^s}$ is given in (175). This can be used in (27), which in turn is to be used in (201).

The bending moment $\overline{M}$ at the specified location is the negative of the sum of the moments about a point $\overline{y}$ in the shear plane due to the internal forces exerted by all the particles of the vehicle on the particles located on one side of the shear plane. In like manner to that employed in determining $\overline{S}$, it is found that

$$\overline{M} = -\sum_i \sum_k (\overline{y}_{ik} - \overline{y}) \times \sum_{\ell=1}^{N} \sum_{j=1}^{P_i} \overline{F}_{ik\ell j}$$

$$= \sum_i \sum_k (\overline{y}_{ik} - \overline{y}) \times \left(\overline{F}_{ik} - m_{ik} \frac{d\overline{v}_{ik}}{dt} \right)$$

$$= \sum_i \sum_k \overline{y}_{ik} \times \overline{F}_{ik} - \sum_i \sum_k m_{ik} \overline{y}_{ik} \times \frac{d\overline{v}_{ik}}{dt}$$

$$- \overline{y} \times \overline{S} \,. \qquad (202)$$

Except for the extent of the summation, the next to the last term of ([illegible]) is like the right side of ([illegible]), which can be used to expand this term for practical use.

$$\bar{S} = \bar{J}_r S_r \quad \text{and} \quad \bar{M} = \bar{J}_r M_r \,. \tag{203}$$

The actual numerical quantities to be computed are the components S_r of $\bar{S}$ and M_r of $\bar{M}$. It is assumed for the purposes of this program that the selected shear plane will be perpendicular to one of the $\bar{J}_r$ vectors. The choice of the shear plane will affect the interpretation of the results S_r and M_r $(r = 1, 2, 3)$. For example, if the shear plane is perpendicular to $\bar{J}_2$, then S_3 is the component of the shear force in the direction of the y^3 axis, S_1 is the shear force in the direction of the y^1 axis, S_2 is the normal force (being perpendicular to the shear plane), M_3 is the bending moment about an axis parallel to y^3, M_1 is the bending moment about an axis parallel to y^1, and M_2 is the torque.

It is a prerequisite to (27) that the sum of the internal forces exerted by and on <u>all</u> the particles of the vehicle equals zero. It is likewise prerequisite to (28) that the sum of the moments about any point in the vehicle due to the internal forces exerted by and on <u>all</u> the particles of the vehicle equals zero. These facts are deduced from Newton's third law of motion; and it follows from these and the definitions of $\bar{S}$ and $\bar{M}$, leading to (201) and (202), that determining $\bar{S}$ and $\bar{M}$ by summing over the opposite side of the shear plane should change their signs but not their magnitudes.

It has been noted in Section 3 that (27) and (28) are not satisfied in the SLP because of the assumption that the elastic deformations and fuel sloshing motions have a negligible effect on the large motions of the vehicle and on $\bar{F}$ and $\bar{G}$. Furthermore, the aerodynamic theory employed here (Newtonian flow) is different from that employed in the SDF program. This difference between the two programs further jeopardizes the agreement between them as to $\bar{F}$ and $\bar{G}$ and, hence, the satisfaction of (27) and (28) in the SLP; therefore, it cannot be expected that summing over the opposite side of the shear plane will satisfy the theoretical requirement of changing only the signs of $\bar{S}$ and $\bar{M}$. This represents a failure to satisfy Newton's third law of motion and may prove to be a serious defect in the program.

[illegible]. INCLUSION OF FUEL SLOSHING EFFECTS

The basic formulations for fuel sloshing as obtained from available literature are presented in Appendix I. Here the concern is how to incorporate the effects of fuel sloshing in the Structural Loads Program.

Within each tank, there are two directions of sloshing, designated somewhat loosely as longitudinal and lateral. We choose the letter u as an indicator of the sloshing direction, longitudinal sloshing being indicated by letting $u=1$, and lateral sloshing being indicated by letting $u=2$.

For each sloshing direction, there are two possible sloshing modes. The letter s is the mode indicator, the first mode being indicated by letting $s=1$, and the second mode being indicated by letting $s=2$.

Since there are two sloshing directions and two possible modes for each direction, there are four possible sloshing degrees of freedom for each tank. The number of the tank is designated by the letter i, and the degree of freedom is designated by k. The following formula is used to determine k in terms of i, u, and s:

$$k = 4(i-1) + u + 2(s-1). \tag{204}$$

The following tabulation illustrates these relations.

i	1				2				3			
s	1		2		1		2		1		2	
u	1	2	1	2	1	2	1	2	1	2	1	2
k	1	2	3	4	5	6	7	8	9	10	11	12

The program allows for a maximum of ten tanks; therefore, the largest possible value of k for a fuel sloshing mode is 40. The number of the elastic degrees of freedom for structural deformation, control surface rotation, and so forth, starts with 41 and proceeds to a maximum of 57, giving a possibility of 17 "structural" degrees of freedom.

The greatest problem that arises in connection with the effects of fuel sloshing in the Structural Loads Program is the computation of the terms H'_{rsi} (138), Λ''^{s}_{ki} (142), and S''^{s}_{jki} (178).

In Appendix I, formulas are given for the effective moments of inertia of the fuel about tank axes for rectangular and cylindrical tanks. The equivalence between these and the H'_{rsi} is as follows, the subscript F denoting <u>fuel</u>:

$$
\left.\begin{aligned}
H'_{r11i} &= J'_{r1i} = I_{rx} \\
H'_{r22i} &= J'_{r2i} = I_{ry} \\
H'_{r33i} &= J'_{r3i} = I_{rz} \\
H'_{rsi} &= 0 \text{ when } r \neq s
\end{aligned}\right\} \text{ for a rectangular tank.} \tag{205}
$$

$$
\left.\begin{aligned}
H'_{r11i} &= J'_{r1i} = 0 \\
H'_{r22i} &= J'_{r2i} = I_{ry} \\
H'_{r33i} &= J'_{r3i} = I_{ry} I_{sz} / I_{sy} \\
H'_{rsi} &= 0 \text{ when } r \neq s
\end{aligned}\right\} \text{ for a horizontal cylindrical tank.} \tag{206}
$$

$$
\left.\begin{aligned}
H'_{r11i} &= J'_{r1i} = m_{ek} \\
H'_{r22i} &= J'_{r2i} = m_{2k} \\
H'_{r33i} &= J'_{r3i} = 0 \\
H'_{rsi} &= 0 \text{ when } r \neq s
\end{aligned}\right\} \text{ for a vertical cylindrical tank.} \tag{207}
$$

All $H'_{rsi} = 0$ for a spherical tank. (208)

In rectangular tanks and for longitudinal sloshing in horizontal cylindrical tanks, a spring-mass mechanical analogy is used. Each mass in this analogy has motion in one, and only one, degree of freedom; therefore, it can be identified by the subscript k, in accordance with equation (204). Likewise, the location of m_k is given by the coordinates x_k, y_k, and z_k. In the case of longitudinal oscillations, $x_k = q^k$ and $y_k = 0$; for lateral oscillations, $x_k = 0$ and $y_k = q^k$; in either case, z_k is simply Z_k, a constant.

For the purposes of evaluating the Λ'^{rs}_{ki} and the S'^{rs}_{ki}, it is necessary to relate the masses and coordinates just discussed with those appearing in (142) and (178). An inspection of (204) quickly discloses that the particular mass particle within tank i is identified by

$$
h = u + 2(s-1), \tag{209}
$$

so that

$$
k = 4(i-1) + h \tag{210}
$$

With this relation between the subscripts i, h, and k established, it is clear that

$$
m_{ih} = m_k \tag{211}
$$

$$
\left.\begin{aligned}
v'_{ih} = x_k &= q^k \quad \text{when } u = 1 \\
&= 0 \quad \text{when } u = 2
\end{aligned}\right\} \tag{212}
$$

$$v^{2}_{\cdot k} = y_k = 0 \text{ when } u = 1 \quad \left.\begin{matrix} \\ = q^{u} \text{ when } u = 2 \end{matrix}\right\} \tag{213}$$

$$v^{3}_{\cdot k} = z_k = \text{constant} \tag{214}$$

From [illegible], $\sigma^{\cdot r}_{u \cdot k} = \partial v^{r}_{\cdot k}/\partial q^{u}$ (215)

Making proper applications now results in the formulas

$$\left.\begin{aligned} \sigma^{\cdot 1}_{u \cdot k} = \partial x_k/\partial q^{u} &= 1 \text{ when } u = 1 \\ &= 0 \text{ when } u = 2 \end{aligned}\right\} \tag{216}$$

$$\left.\begin{aligned} \sigma^{\cdot 2}_{u \cdot k} = \partial y_k/\partial q^{u} &= 0 \text{ when } u = 1 \\ &= 1 \text{ when } u = 2 \end{aligned}\right\} \tag{217}$$

$$\sigma^{\cdot 3}_{u \cdot k} = \partial z_k/\partial q^{u} = 0 \tag{218}$$

Summarizing (212) thru (218) results in

$$\left.\begin{aligned} v^{\cdot r}_{\cdot k} &= q^{u} \text{ when } r = u \\ &= 0 \text{ when } r \neq u \end{aligned}\right\} r \neq 3 \tag{219}$$

$$= z_k \text{ when } r = 3 \tag{220}$$

$$\left.\begin{aligned} \sigma^{\cdot t}_{u \cdot k} &= 1 \text{ when } t = u \\ &= 0 \text{ otherwise} \end{aligned}\right\} \tag{221}$$

Substitution from (211), (219), (220), and (221) into (142) and letting the unknown $q^{u} = 0$ for this purpose results in

$$\begin{aligned} \Lambda^{rt}_{ku} &= m_k z_k \text{ when } r = 3 \text{ and } t = u \\ &= 0 \text{ otherwise} \end{aligned} \tag{222}$$

Similar substitution into (178) results in

$$\begin{aligned} \delta^{\cdot rs}_{jkl} &= m_k \text{ when } r = s = u \\ &\qquad \text{and } j = k \\ &= 0 \text{ otherwise} \end{aligned} \tag{223}$$

The reader is reminded that (222) and (223) are applicable only to rectangular tanks and to longitudinal sloshing in horizontal cylindrical tanks. For other tanks, the $\Lambda^{\cdot rt}_{ku}$ are assumed to be non-existent, and the $\delta^{\cdot rs}_{jkl}$ are more or less circumvented by arriving at the μ_{jk} (which equal $\sum_{l=1}^{4} \delta^{\cdot rr}_{jkl}$) by another process.

The formulation presented in Appendix I for lateral oscillations in horizontal cylindrical tanks, for vertical cylindrical tanks, and for spherical tanks lead to expressions for the kinetic and potential energies of fuel sloshing rather than a spring-mass mechanical analogy. Once these expressions for the kinetic energy are extended to account for the other motions of the vehicle as well as the sloshing of the fuel, they can be used as in (72) to obtain expressions for the contribution of the fuel to the M_{jk}.

To lateral sloshing in a horizontal cylindrical tank we affix the subscript 1 to T, M_F, ρ, and a and make the following substitutions in the final equation given in Appendix I for the kinetic energy

$$\left.\begin{aligned} \dot{q}' &= v_1'^{e} \\ \ell &= b_{n1} \\ A_{s+1} &= A_{s1} \\ B_{s+1} &= B_{s1} \\ \left[\sqrt{\lambda_{s+1}}\right]^2 &= \lambda_{s1} \end{aligned}\right\} \tag{224}$$

In addition to this, we introduce a transformation of coordinates. Let

$$p = 4(\;-1) + u \,, \tag{225}$$

then (204) can be expressed

$$k = p + 2(s-1) \tag{226}$$

and we let

$$q^{s+2} = \frac{R_{n1}}{\lambda_{s1}}\, q^{p+2(s-\;)} \tag{227}$$

This results in

$$T_1 = \frac{1}{2} M_{F1} \left(v_1'^{e}\right)^2 + \rho_1 a_1 b_{n1} R_{n1}^2 \sum_{s=1}^{\infty} \frac{A_{s1}}{\lambda_{s1}} \left(\dot{q}^{p+2(s-1)}\right)^2 + 2 \rho_1 a_1^2 b_{n1} R_{n1}^2 v_1'^{e} \sum_{s=1}^{\infty} B_{s1} q^{p+2(s-1)} \tag{228}$$

Differentiation of T_1 results in

$$\frac{\partial T_1}{\partial q^{p+2(s-1)}} = 2 \rho_1 a_1 b_{n1} R_{n1}^2 \left(A_{s1}/\lambda_{s1}\right) \dot{q}^{p+2(s-1)} + 2 \rho_1 a_1^2 b_{n1} R_{n1}^2 v_1'^{e} B_{s1} \tag{229}$$

$$\frac{\partial^2 T_1}{\partial q^{r+2(s-1)}\,\partial q^{p+2(s-1)}} = 2\rho_1 a_1 b_{H1} R_{H1}^2 A_{s1}/\lambda_{s1} \qquad (230)$$

$$\frac{\partial^2 T_1}{\partial q^j\,\partial q^{p+2(s-1)}} = 2\rho_1 a_1^2 b_{H1} R_{H1}^2 B_{s1}\, \ell'_{j1} h'_{j1} \qquad (231)$$

j denoting a "structural" (that is, non-fuel slosh) degree of freedom, and ℓ'_{j1}, h'_{j1} being demonstrably equal to $\frac{\partial}{\partial q^j}$ by (62) (120), in recognition that here $\bar{\sigma}_H$ and $\bar{\upsilon}$ equal zero.

From (230), for the case in which j and k denote lateral sloshing in a horizontal cylindrical tank, we define

$$\begin{aligned} \mu_{jk} &= 2\rho_1 a_1 b_{H1} R_{H1}^2 A_{s1}/\lambda_{s1} \\ &\qquad \text{when } j = k = p + 2(s-1) \\ &= 0 \quad \text{when } j \neq k \;. \end{aligned} \qquad (232)$$

From (231), for the case in which k denotes lateral sloshing in a horizontal cylindrical tank, we define

$$m'_k = 2\rho_1 a_1^2 b_{H1} R_{H1}^2 B_{s1} \qquad (233)$$

For sloshing in a spherical tank, we affix the subscript 1 to T, M_F, ρ, a, and R and make the following substitutions in the final equation given in Appendix I for the kinetic energy:

$$\begin{aligned} q^r &= \upsilon_1'^r \quad (r = 1 \text{ or } 2) \\ C_{s+3} &= C_s \\ D_{s+3} &= D_{s1} \\ [\sqrt{\lambda_{s+3}}\,]^2 &= \lambda'_{s1} \end{aligned} \qquad (234)$$

In addition to this, we use (225) and (226) again and introduce the following transformation of coordinates:

$$q^{s+3} = \frac{R_1}{\lambda'_{s1}}\, q^{p+2(s-1)} \qquad (235)$$

This results in

$$\begin{aligned} T_1 &= \tfrac{1}{2} M_{F1} (\upsilon_1'^r)^2 + \tfrac{1}{2}\pi \rho_1 a_1^2 R_1^3 \sum_{s=1}^{\infty} \frac{C_{s1}}{\lambda'_{s1}} (q^{p+2(s-1)})^2 \\ &\quad + \pi \rho_1 a_1^2 R_1^3 \upsilon_1'^r \sum_{s=1}^{\infty} D_{s1}\, \dot{q}^{p+2(s-1)} \end{aligned} \qquad (236)$$

Differentiation of T_i results in

$$\frac{\partial T_i}{\partial \dot{q}^{\,p+2(s-1)}} = \pi \rho_i a_i^2 R_i^3 \frac{C_{si}}{\lambda'_{si}} \dot{q}^{\,p+2(s-1)} + \pi \rho_i a_i^3 R_i^3 v_i'^{\,r} D_{si} \tag{237}$$

$$\frac{\partial^2 T_i}{\partial \dot{q}^{\,p+2(s-1)} \partial \dot{q}^{\,p+2(s-1)}} = \pi \rho_i a_i^2 R_i^3 C_{si} / \lambda'_{si} \tag{238}$$

$$\frac{\partial^2 T_i}{\partial \dot{q}^{\,j} \partial \dot{q}^{\,p+2(s-1)}} = \pi \rho_i a_i^3 R_i^3 D_{si} \ell_{ri}^{t} h_{ji}^{t}, \tag{239}$$

j denoting a structural degree of freedom.

From (238), for the case in which j and k denote sloshing in a spherical tank, we define

$$\begin{aligned} \mu_{jk} &= \pi \rho_i a_i^2 R_i^3 C_{si} / \lambda'_{si} \\ &\qquad \text{when } j = k = p + 2(s-1) \\ &= 0 \quad \text{when } j \neq k \end{aligned} \tag{240}$$

From (239), for the case in which k denotes sloshing in a spherical tank, we define

$$m'_k = \pi \rho_i a_i^3 R_i^3 D_{si} \tag{241}$$

Making use of the m'_k from either (233) or (241), we compute for a spherical tank or for lateral sloshing in a horizontal cylindrical tank

$$\phi_{jk} = \ell_{ri}^{t} (h_{ji}^{t} m'_k + h_{ki}^{t} m'_j). \tag{242}$$

The equivalence of this to (231) or (239) should be noted.

For sloshing in a vertical cylindrical tank, we note from the given equation for T in Appendix I that

$$\frac{\partial^2 T}{\partial \dot{q}^{\,l} \partial \dot{q}^{\,p}} = m_{lp}. \tag{243}$$

Now q^{l} and q^{p} need to be related to the coordinates for measurement of the structural deflections and the sloshing of the fluid in order to determine expressions for the M_{jk}. For this purpose, we employ Equation (2-16) from Reference (7). Putting this equation into the terms that are appropriate to the present purpose results in

$$M_{jk} = m_{ip} \frac{\partial q^i}{\partial q^j} \frac{\partial q^p}{\partial q^k} \tag{244}$$

In determining the partial derivatives of (244), we distinguish between "structural" and "fuel slosh" degrees of freedom as before. Thus, for structural degrees of freedom (assuming $u = 2$, that is, that the motion is in the ℓ_1, ℓ_2 plane).

$$\left.\begin{aligned} &\frac{\partial q^1}{\partial q^j} = \ell^r_{2i} h^r_{ji} , \quad \frac{\partial q^2}{\partial q^j} = \alpha'_{ji} , \\ &\frac{\partial q^3}{\partial q^j} = 1 \text{ when } 3 \text{ is replaced by } j \\ &\frac{\partial q^{s+3}}{\partial q^j} = 0 \end{aligned}\right\} \tag{245}$$

For fuel sloshing degrees of freedom

$$\left.\begin{aligned} &\frac{\partial q^1}{\partial q^{p+2(s-1)}} = 0 , \quad \frac{\partial q^2}{\partial q^{p+2(s-1)}} = 0 \\ &\frac{\partial q^3}{\partial q^{p+2(s-1)}} = 0 , \\ &\frac{\partial q^{s+3}}{\partial q^{p+2(s-1)}} = 1 \text{ when } s+3 \text{ is replaced by } p+2(s-1) \end{aligned}\right\} \tag{246}$$

The following substitutions are also made:

$$\begin{aligned} &m_{11} = M_{F1} , \quad m_{13} = u''_{1k1} , \quad m_{1,s+3} = u''_{1,\,p+2(s-1)} , \\ &m_{22} = J'_{F11} , \quad m_{23} = u''_{2k1} , \quad m_{2,s+3} = u''_{2,\,p+2(s-1)} ; \\ &m_{33} = u'_{kk1} , \quad m_{3,s+3} = u'_{k,\,p+2(s-1)} ; \\ &m_{s+3,s+3} = u'_{p+2(s-1),\,p+2(s-1)} \end{aligned} \tag{247}$$

Finally, substitution from (245), (246), and (247) into (244) results in

$$\begin{aligned} M_{jk} = {} & M_{F1} \ell^r_{2i} h^r_{ji} \ell^s_{2i} h^s_{ki} + u''_{1k1} \ell^r_{2i} h^r_{ji} \\ & + J'_{F11} \alpha'_{ji} \alpha'_{ki} + u''_{2k1} \alpha'_{ji} + u''_{1j1} \ell^r_{2i} h^r_{ki} \\ & + u''_{2j1} \alpha'_{ki} + u'_{jk1} , \end{aligned} \tag{248}$$

$$\begin{aligned} M_{j,\,p+2(s-1)} = {} & u''_{1,\,p+2(s-1)} \ell^r_{2i} h^r_{ji} + u''_{2,\,p+2(s-1)} \alpha'_{ji} \\ & + u'_{j,\,p+2(s-1)} = \phi_{jk1} , \end{aligned} \tag{249}$$

$$M_{\mu+2(s-1)\,\mu+2(s-1)} = \mu_{\mu+2(s-1)\,\mu+2(s-1)} = \mu_{jk}$$
$$= 0 \text{ when } j \neq k \qquad (250)$$

For all sloshing in spherical and vertical cylindrical tanks, and lateral sloshing in horizontal tanks, there is no dynamic balancing.

12. POINT OF ROTATION AND DYNAMIC BALANCING OF CERTAIN DATA FOR FLEXIBLE SECTIONS

Within every section is a point p_i (with sectional position vector $\bar{p}$ and vehicle position vector $\bar{x}_i$) which may or may not coincide with the center of mass of the section. Since $\bar{\sigma}_i$ is the vehicle position vector of the center of mass of section i,

$$\bar{x}_i = \bar{\sigma}_i + \bar{p}_i \tag{251}$$

The point p_i is fixed in the physical material of the section and moves with it when and if it moves. Thus, it may correspond with a particle of the section.

If the section is "movable", that is, has motion of type (2), then p_i is a point of rotation of the section - a point in the section that does not move relative to the vehicle but about which the section rotates in a type (2) motion. If the section rotates about a fixed axis, p_i lies somewhere on this axis.

As for motions of type (3), the unbalanced motion of p_i relative to the vehicle coordinates is given prior to that of any other point in the section. If the degree of freedom deforms the section, and if the section is movable, p_i does not move relative to the vehicle in that degree of freedom. On the other hand, if the section is "fixed", or if the degree of freedom does not deform the section, p_i may move relative to the vehicle coordinates in that degree of freedom.

From (130), (135), (136), and (22),

$$\sum_{h=1}^{P_i} m_{ih}\,\bar{a}_{kih} = \sum_{h=1}^{P_i} m_{ih}\,\bar{\gamma}_{ki} = m_i\,\bar{\gamma}_{ki}\,, \tag{252}$$

$$\bar{\gamma}_{ki} = \frac{1}{m_i}\sum_{h=1}^{P_i} m_{ih}\,\bar{a}_{kih}\,, \tag{253}$$

$$\bar{\sigma}_{kih} = \bar{a}_{kih} - \bar{\gamma}_{ki} - \bar{\beta}_{ki} \times \bar{v}_{ih} \tag{254}$$

Here the $\bar{a}_{kih}$ and the $\bar{\beta}_{ki}$ are given arbitrarily, and the $\bar{\gamma}_{ki}$ and $\bar{\sigma}_{kih}$ are determined from them. This is necessary for the satisfaction of (136) when the degree of freedom (k) deforms the section (i).

When $\bar{v}_{ih} = \bar{p}_i$, then $\bar{y}_{ih} = \bar{x}_i$, $\bar{a}_{kih} = \bar{a}_{ki}$, $\bar{\sigma}_{kih} = J_{si}\dfrac{\partial \bar{p}'^{i}}{\partial q^{k}} = \bar{e}_{ki}$, and we introduce

$$\bar{x}_{ki} = \bar{a}_{ki} = \bar{e}_{ki} + \bar{\gamma}_{ki} + \bar{\beta}_{ki} \times \bar{p}_i \tag{255}$$

When the degree of freedom (k) does not deform the section (i), then $\bar{\sigma}_{kih} = 0$, $\bar{\rho}_{ki} = 0$, (136) is still satisfied, and $\bar{\alpha}_{ki} = \hat{a}_{ki}$.

When the section is movable and is deformed by the degree of freedom, determine the α'^{r}_{kih} and the β'^{r}_{ki}. (The β'^{r}_{ki} are arbitrary.) Then compute and submit

$$\gamma'^{r}_{ki} = \frac{1}{\overline{m}_i} \sum_{h=1}^{n} m_{ih}\, \alpha'^{r}_{kih} \tag{256}$$

$$\sigma'^{r}_{kih} = \alpha'^{r}_{kih} - \gamma'^{r}_{ki} - C_{rsc}\, \beta'^{s}_{ki}\, v^{c}_{ih} \tag{257}$$

The $\alpha'^{r}_{ki} = 0$ in this case. This fact effects the location of p_i, the point of rotation. If the section is fixed and deformed, p_i can be arbitrarily located and the α'^{r}_{ki} are unimportant.

When the section is movable and is deflected but not deformed by the degree of freedom, determine and submit α^{r}_{ki} and β^{r}_{ki} (without primes).

The symbols, data to be submitted, computations and equations used in the SLP are given in Appendix II.

13. REFERENCES

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APPENDIX I

BASIC FORMULATIONS FOR FUEL SLOSHING

INTRODUCTION

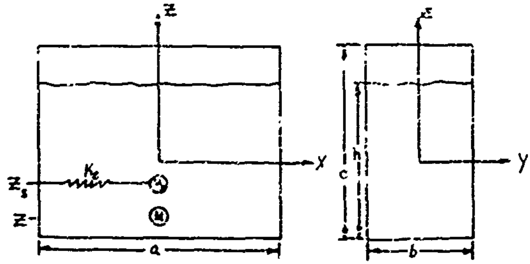
For sufficiently small amplitudes of motion, the dynamic effects of the sloshing of fuel in a partially filled tank have been analyzed in terms of the natural modes and frequencies of the small, free-surface oscillations of the fuel. Formulations for the solution of this problem are presented for rectangular, cylindrical, and spherical tanks. In Part I of this Appendix, natural modes and frequencies are presented for rectangular and cylindrical tanks in either a vertical or a horizontal position. In Part II, an approximate procedure is established for dealing with rectangular and cylindrical tanks that are neither vertical nor horizontal. Part III has to do with the treatment of the bending mode shape of an equivalent vertical tank. Part IV of this Appendix concerns the inclusion of fuel damping in the SLP.

PART I

The formulas presented in this part for rectangular and cylindrical tanks apply only to vertical or horizontal positions. Those presented for spherical tanks do not need to be qualified as to position.

In order to simplify the problem to a point where convenient, explicit solutions could be obtained in most cases, a number of assumptions were made concerning the nature of the fuel, the motions of the fuel and the shape of the tank. The fuel was assumed to be non-viscous and incompressible and all tank motions, except those normal to the mean free surface of the fuel, were restricted to small accelerations and perturbations. Although the non-viscous assumption has been made, a damping factor will be included in the final SDF Structural Loads Program to account for the fuel viscosity and the use of baffles. It should also be noted that, in the final program, provisions are made for summing any combination of rectangular, cylindrical or spherical tanks for multiple tank vehicles.

1. Rectangular Tank - In the case of a rectangular tank, a spring mass mechanical [illegible] The [illegible] below are for a system consisting of a fixed mass M and a set of undamped spring-masses m_s, so constrained as to move only parallel to the bottom of the tank and, in the case of the horizontal rectangular tank, parallel to the XZ-plane. The origin of the axes is located at the center of gravity of the undisturbed fuel with the fixed mass M located Z_0 and the spring-masses at Z_s constrained by springs with stiffness K_s for the s^{th} mode. Shown below is the horizontal rectangular tank undergoing longitudinal oscillations, or oscillations in the XZ-plane. The motion in the XZ-plane is assumed to be the same regardless of the Y location. The following equations have been developed by Graham in Reference (1).



Definitions:

a	= tank length parallel to X-axis
b	= tank width parallel to Y-axis
c	= tank height parallel to Z-axis
h	= fuel height parallel to vertical axis
M_F	= ρabh = total fuel mass
s	= fuel mode index = 1, 2, 3, --- ω. ω denotes the number of modes selected for use.
ρ	= fuel density
g	= acceleration of tank normal to mean free surface of fuel
W_F	= gM_F = total fuel weight
r_1	= h/a = tank aspect ratio
I_{sy}	= moment of inertia about Y-axis if the fuel were solidified
I_{fy}	= effective moment of inertia about the Y-axis
ω_s	= frequency of the s^{th} mode of free surface oscillation

Equations:

$$\omega_s = \left[g(2s-1)\frac{\pi}{a}\,\mathrm{TANH}[(2s-1)\pi r_1]\right]^{1/2}$$

$$m_s = M_F\,\frac{8\,\mathrm{TANH}[(2s-1)\pi r_1]}{\pi^3(2s-1)^3 r_1}$$

$$K_s = \frac{8W_F\,\mathrm{TANH}^2[(2s-1)\pi r_1]}{h\pi^2(2s-1)^2}$$

$$M = M_F - M_F \sum_{s=1}^{\infty} \frac{8\,\mathrm{TANH}[(2s-1)\pi r_1]}{\pi^3(2s-1)^3 r_1}$$

$$Z_s = \frac{h}{2} - \frac{h\,\mathrm{TANH}[(2s-1)\frac{\pi}{2} r_1]}{(2s-1)\frac{\pi}{2} r_1}$$

$$Z = -\frac{1}{M} \sum_{s=1}^{\infty} m_s Z_s$$

$$I_{FY} = I_{SY}\left\{1 - \frac{4}{1+r_1^2} + \frac{768}{r_1(1+r_1^2)\pi^5} \sum_{s=1}^{\infty} \frac{\mathrm{TANH}[(2s-1)\frac{\pi}{2} r_1]}{(2s-1)^5}\right\}$$

$$I_{SY} = M_F \frac{a^2+h^2}{12}$$

In the case of a horizontal rectangular tank undergoing lateral oscillations in the YZ-plane the definitions and equations are unchanged except that the moments of inertie are now I_{FX} and I_{SX} and the tank aspect ratio now becomes $r_2 = h/b$ with

$$\omega_s = \left[g(2s-1)\frac{\pi}{b}\,\mathrm{TANH}[(2s-1)\pi r_2]\right]^{1/2} \quad \text{and} \quad I_{SX} = \frac{b^2+h^2}{12} M_F$$

If the tank is now rotated so that the X-axis is vertical with the fuel oscillating parallel to the XZ-plane, the value of $M_F = \rho cbh$ and the tank aspect ratio becomes $r_3 = h/c$. The moments of inertia are still taken about the y-axis and the equations for Z_s and Z are the same except they become X_s and X distances and use r_3 instead of r_1. The equations for ω_s and I_{sy} become:

$$\omega_s = \left[g(2s-1)\frac{\pi}{c}\,\mathrm{TANH}[(2s-1)\pi r_3]\right]^{1/2}$$

$$I_{SY} = M_F \frac{c^2+h^2}{12}$$

With the X-axis vertical but with the oscillations parallel to the XY-plane, $M_F = \rho cbh$, the tank aspect ratio $r_4 = h/b$, the moments of inertia are I_{FZ} and I_{SZ} and the equations for ω_s and I_{SZ} become:

$$\omega_s = \left[g(2s-1)\frac{\pi}{b}\,\mathrm{TANH}[(2s-1)\pi r_4]\right]^{1/2}$$

$$I_{SZ} = M_F \frac{b^2+h^2}{12}$$

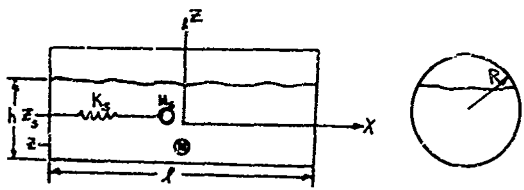
This completes the specification of the equations for the spring-mass analogy for rectangular tanks in a horizontal or vertical orientation. Therefore, the angle the X-axis makes with the horizontal determines which set of equations more accurately approximates the situation.

2. Cylindrical Tank - The formulations for the cylindrical tank are not nearly as straight forward as were those for the rectangular tanks. Three different methods have been used to define the fuel motion for the different

tank orientations. An extensive literature survey indicated that for the case of longitudinal oscillations in a horizontal cylindrical tank the same approach could be used as in the case of rectangular tanks although no development of the equations could be found. Reference (2) suggested that the natural frequencies for the horizontal cylindrical tank are:

$$\omega_s = \left[\frac{g s\pi}{l}\, TANH\, \frac{s\pi h}{l}\right]^{1/2} \qquad \text{where } s = 1, 2, 3, \text{---} \infty$$

Comparing this equation with the corresponding frequency equation for rectangular tanks indicates that the cylindrical tank aspect ratio is r = h/l. Making like comparisons the following development is suggested.



Definitions:

l	=	tank length parallel to X-axis
R	=	tank radius
h	=	fuel height parallel to vertical axis
M_f	=	total fuel mass
s	=	1, 2, 3, --- ∞
r	=	h/l = tank aspect ratio

Equations:

$$\omega_s = \left[\frac{g s\pi}{l}\, TANH(s\pi r)\right]^{1/2}$$

$$m_s = M_f \frac{8\, TANH(s\pi r)}{\pi^3 s^3 r}$$

$$K_s = \frac{8 W_f\, TANH^2(s\pi r)}{h \pi^2 s^2}$$

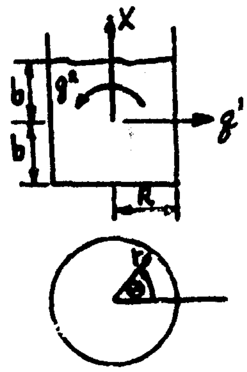
$$M = M_f - M_f \sum_{s=1}^{\infty} \frac{8\, TANH(s\pi r)}{\pi^3 s^3 r}$$

$$Z_s = \frac{h}{2} - \frac{h\,TANH(\frac{1}{2}s\pi r)}{\frac{1}{2}s\pi r}$$

$$Z = -\frac{1}{M}\sum_{s=1}^{\infty} m_s Z_s$$

$$I_{FY} = I_{sY}\left\{1 - \frac{4}{1+r^2} + \frac{768}{r(1+r^2)\pi^6}\sum_{s=1}^{\infty}\frac{TANH(\frac{1}{2}s\pi r)}{s^5}\right\}$$

The second method to be used on the cylindrical tank follows the formulations of J. W. Miles found in Reference (3) for an upright circular cylinder. In this analysis the potential and kinetic energy expressions are derived with allowances made for tank flexibility. First the potential energy (U) and the kinetic energy (T) expressions are stated and then the potential energy coefficients (k_{ij}) and the inertia coefficients (m_{ij}) are defined.



Definitions:

$i = j = s = 1, 2, 3, \cdots \infty$

$q^i(t)$ = generalized coordinates

$q^1(t)$ = a translation along $\theta = 0$

$q^2(t)$ = a rotation about the centroidal axis $\theta = \frac{\pi}{2}$

$q^3(t)f(X)$ = a simple bending displacement along $\theta = 0$

$q^{s+3}\psi_{s+3}(r,\theta)$ = sloshing displacements

$f(X)$ = bending mode shape of tank

$f'(X) = df(X)/dx$

$\psi_{s+3}(r,\theta)$ = s^{th} mode shape of fuel

$M = 2\pi\rho R^2 b$ = total mass of fuel

Definitions (Continued)

- s = index indicating fuel slosh modes
- b = one half fluid height
- R = tank radius
- g = acceleration of tank along X-axis
- β_s = s^{th} zero of the first derivative of the Bessel Function of the first order and the first kind. (β_1 = 1.84119, β_2 = 5.33144, β_3 = 8.53631, β_4 = 11.70600)

δ_j^i = 1 if i = j
0 if i ≠ j

Equations

$$U = \frac{1}{2}\sum_i \sum_j k_{ij}\, q^i q^j$$

Reference (3) defines the potential energy coefficients, k_{ij} as shown below:

$$k_{11} = k_{12} = k_{21} = k_{13} = k_{31} = k_{22} = k_{23} = k_{32} = k_{1,s+3}$$

$$= k_{s+3,1} = 0$$

$$k_{33} = Mg\left\{\frac{R^2}{8b}\left[f'^2(b) - f'^2(-b)\right] + \frac{1}{2b}\int_{-b}^{b} x f'^2(x)\,dx - \frac{1}{2}\int_{x_0}^{b} f^2(x)\,dx + \frac{1}{2}\int_{-b}^{x_0} f'^2(x)\,dx\right\}$$

$$k_{s+3,s+3} = \frac{1}{4}\frac{Mg}{b\beta_s^2}\left(\beta_s^2 - 1\right)$$

$$k_{2,s+3} = k_{s+3,2} = \frac{1}{2}\frac{MgR}{b\beta_s^2}$$

$$k_{3,s+3} = k_{s+3,3} = -\frac{1}{2}\frac{MgR}{b\beta_s^2} f'(b)$$

$$T = \frac{1}{2}\sum_i \sum_j m_{ij}\, \dot{q}^i \dot{q}^j$$

[illegible] coefficients, m_{ij}, are de[illegible]

$$m_{11} = M$$

$$m_{[illegible]} = m_{[illegible]} = 0$$

$$m_{12} = m_{21} = M\left\{F_0 - \frac{R^2}{8b}\left[f'(b) - f'(-b)\right]\right\}$$

$$m_{1,s+3} = m_{s+3,1} = \frac{MR}{2b\beta_s^2}$$

$$m_{22} = M\left(\frac{b^2}{3} - \frac{3R^2}{4}\right) + \frac{8MR^3}{b}\sum_{s=1}^{\infty}\frac{\mathrm{TANH}\left(\frac{\beta_s b}{R}\right)}{\beta_s^2(\beta_s^2-1)}$$

$$m_{23} = m_{32} = +M\left\{\frac{1}{2b}\int_{-b}^{b} x f(x)\,dx + \frac{8b}{\pi^2}\sum_{s=1}^{\infty}\frac{\psi_{2s-1}F_{2s-1}}{(2s-1)^2}\right.$$

$$+\left[f'(b)+f'(-b)\right]\left[\frac{b^2}{3} - \frac{R^2}{8} + \frac{32b^2}{\pi^4}\sum_{s=1}^{\infty}\frac{\psi_{2s-1}}{(2s-1)^4}\right.$$

$$m_{2,s+3} = m_{s+3,2} = \frac{MR}{2\beta_s^2}\left[\frac{2R\,\mathrm{TANH}\left(\frac{\beta_s b}{R}\right)}{\beta_s b} - 1\right]$$

$$m_{33} = MF_0^2 + \frac{M}{2}\sum_{s=1}^{\infty}\psi_s F_s^2 - \frac{MR^2}{b}\sum_{s=1}^{\infty}\frac{f'(b)Y_s(b) - f'(-b)Y_s(-b)}{\beta_s^2-1}$$

$$+ \frac{F_0 MR^2}{8b}\left[f'(-b) - f'(b)\right] + \frac{2[illegible]}{\pi^{[illegible]}}\sum_{s=1}^{\infty}\frac{1-\psi_s}{s^2}\left[(-)^{s+1}f'(b) + f'(-b)\right]F_s$$

$$+ \frac{[illegible]^3}{b}\sum_{s=1}^{\infty}\frac{[illegible]\,\frac{2\beta_s b}{R}}{\beta_s^3(\beta_s^2-1)}\left\{\left[f'^2(b) + f'^2(-b)\right]\cosh\left(\frac{2\beta_s b}{R}\right)\right.$$

$$\left. - 2f'(b)f'(-b)\right\}$$

$$m_{3,s+3} = m_{s+3,3} = \frac{MR}{2b}Y_s(b) - \frac{MR^2}{2b\beta_s^2}\left[\mathrm{CSCH}\left(\frac{2\beta_s b}{R}\right)\right]\left[f'(b)\cosh\left(\frac{2\beta_s b}{R}\right) - f'(-b)\right]$$

$$m_{s+3,s+3} = \frac{MR}{4bA_s^2}\left(\beta_s^2 - 1\right) \coth\left(\frac{2\beta_s b}{R}\right)$$

where $A'_s = \frac{S\pi R}{2b}$

$$\psi(0) = 1.0$$

$$\psi_s = \psi(A'_s) = 1 - \left[\frac{I_2(A'_s)}{I_1(A'_s)}\right]$$

I'_1 and I_2 are certain Bessel functions or certain forms of functions of some related kind.

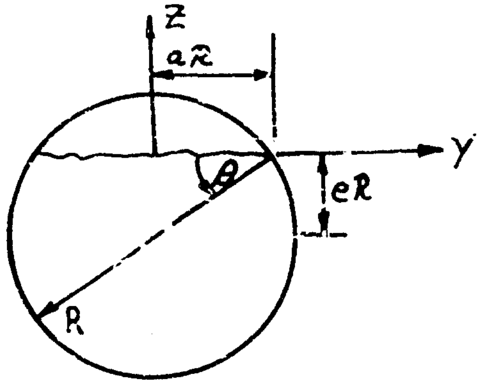
$$F_s = \left(\frac{2}{2b}\right)\int_{-b}^{b} f(x) \cos\left[s\pi\left(\frac{x+b}{2b}\right)\right] dx \quad \text{when} \quad S > 0 .$$

$$F_0 = \frac{1}{2b}\int_{-b}^{b} f(x)\,dx$$

$$Y_s(b) = \sum_{p=0}^{\infty} (-1)^p \left[\beta_s^2 + \left(\frac{p\pi R}{2b}\right)^2\right]^{-1} F_p$$

These equations have been used in the experimental analysis of Reference (4). In this report, resonant bending frequencies and mode shapes were determined experimentally and were shown to be generally in agreement with the oretical predictions. The differences were attributed primarily to variations of actual mode shapes from those assumed in the theory.

The third method to be used on cylindrical tanks was formulated by B. Budiansky in Reference (5). This is also the method to be used for spherical tanks. In this report an integral equation approach is used and the method of solution developed for the first three fuel slosh mode , which as indicated in the literature is a sufficient number of modes for most practical problems. The tank orientation under consideration is a horizontal cylindrical tank undergoing lateral oscillations. In this case the generalized coordinate denoting motion along the Y-axis is q^1 and again the fuel sloshing generalized coordinates are q^{s+3}. The dimensions of the q^{s+3} in this development, however, are $(\text{length})^2$ rather than length as in the case of the previous generalized coordinates. The slosh height, $\mathcal{f}^{s+3}$, at the side of the tank can be expressed as a function of the q^{s+3} by the following relation, $\mathcal{f}^{s+3} = q^{s+3}\omega^{s+3}/g$. It should be noted that in this analysis tank bending is ignored and that with the non-viscous assumption rotation of this cylindrical tank and rotation of the spherical tank need not be considered.



Definitions:

q^1	= a translation along the Y-axis
R	= tank radius
e	= fuel height parameter = -1.0 for empty tank
	= $\sin\beta$ = 0 for half full tank
	= +1.0 for full tank
β	= $\sin^{-1} e$
a	= $\cos\beta$
λ_{s+3}	= frequency parameter = $\frac{R}{g}\omega^2_{s+3}$
s	= fuel slosh mode index = 1, 2, 3, --- ∞
ℓ	= tank length
ρ	= fuel density
g	= tank acceleration along Z-axis
M_F	= total mass of fuel

Equations:

$$U = \frac{\rho a R \ell}{g} \sum_{s=1}^{\infty} \omega^4_{s+3} A_{s+3} (q^{s+3})^2$$

$$U = \frac{\rho a g \ell}{R} \sum_{s=1}^{2} \left[\sqrt{\lambda_{s+3}}\right]^4 A_{s+3} (q^{s+3})^2$$

$$T = \frac{1}{2} M_F (\dot{q}^1)^2 + \frac{\rho a R \ell}{g} \sum_{s=1}^{\infty} \omega^2_{s+3} A_{s+3} (\dot{q}^{s+3})^2 + \frac{2\rho \ell}{g} (aR)^2 \dot{q}^1 \sum_{s=1}^{\infty} \omega^2_{s+3} B_{s+3} \dot{q}^{s+3}$$

$$T = \frac{1}{2} M_F (\dot{g}')^2 + \rho a \ell \sum_{s=1}^{3} \left[\sqrt{\lambda_{s+3}}\right]^2 A_{s+3} (\dot{g}^{s+3})^2$$

$$+ 2 \rho a^2 R \dot{g}' \ell \sum_{s=1}^{3} \left[\sqrt{\lambda_{s+3}}\right]^2 B_{s+3} \dot{g}^{s+3}$$

The nondimensional modal parameters A_{s+3} and B_{s+3} along with $\sqrt{\lambda_{s+3}}$ are presented in Figures (4), (5), and (6) respectively for the first three fuel slosh modes as functions of the fuel height parameter e. It should be noted that for values of e = .8 to e = 1.0, the curves in Figures (6) and (9) tend to infinity as they approach e = 1.0. To avoid this type of solution near e = 1.0, the curves have been made to intersect e = 1.0 to provide for an approximate but finite solution for the full tank. For this reason the solution of the equations for the nearly full to full cylindrical and spherical tanks must be used with caution.

3. <u>Spherical Tanks</u> - Following the same method and definitions as used above, the solutions for the spherical tank may be obtained.

Equations

$$U = \frac{\pi \rho}{2 g} (aR)^2 \sum_{s=1}^{\infty} \omega_{s+3}^4 C_{s+3} (g^{s+3})^2$$

$$U = \frac{1}{2} \pi \rho g a^2 \sum_{s=1}^{3} \left[\sqrt{\lambda_{s+3}}\right]^4 C_{s+3} (g^{s+3})^2$$

$$T = \frac{1}{2} M_F (\dot{g}')^2 + \frac{\pi \rho}{2 g} (aR)^2 \sum_{s=1}^{\infty} \omega_{s+3}^2 C_{s+3} (\dot{g}^{s+3})^2$$

$$+ \frac{\pi \rho}{g} (aR)^3 \dot{g}' \sum_{s=1}^{\infty} \omega_{s+3}^2 D_{s+3} \dot{g}^{s+3}$$

$$T = \frac{1}{2} M_F (\dot{g}')^2 + \frac{1}{2} \pi \rho a^2 R \sum_{s=1}^{3} \left[\sqrt{\lambda_{s+3}}\right]^2 C_{s+3} (\dot{g}^{s+3})^2$$

$$+ \pi \rho a^2 R^2 \dot{g}' \sum_{s=1}^{3} \left[\sqrt{\lambda_{s+3}}\right]^2 D_{s+3} \dot{g}^{s+3}.$$

As before, the values of the nondimensional modal parameters C_{s+3} and D_{s+3} along with $\sqrt{\lambda_{s+3}}$ are plotted versus e in Figures (7), (8), and (9) respectively. As discussed previously, the solutions of the equations are only approximate solutions as the full condition is approached.

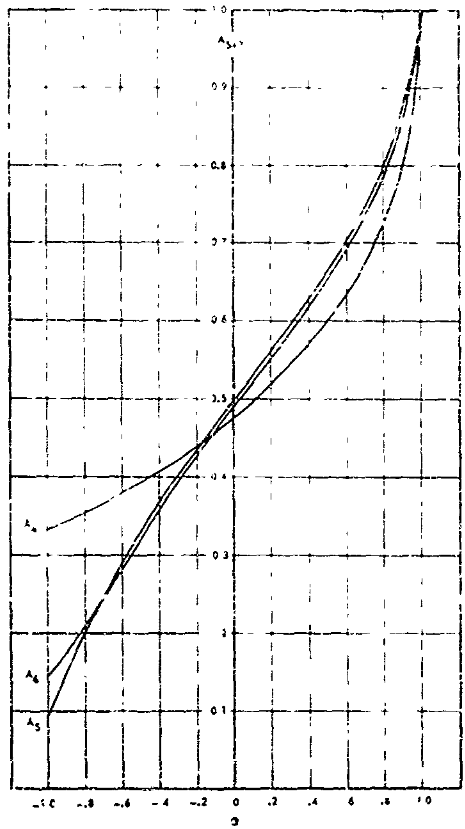


Figure 4. Variation of A_{5+3} with Fuel Height Parameter, e, Cylindrical Tank

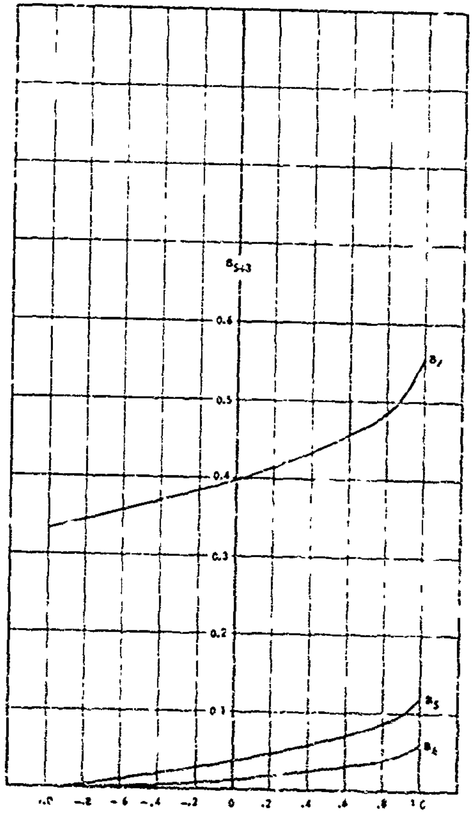


Figure 5 Variation of B_{5+3} with Fuel Height Parameter, e, Cylindrical Tank

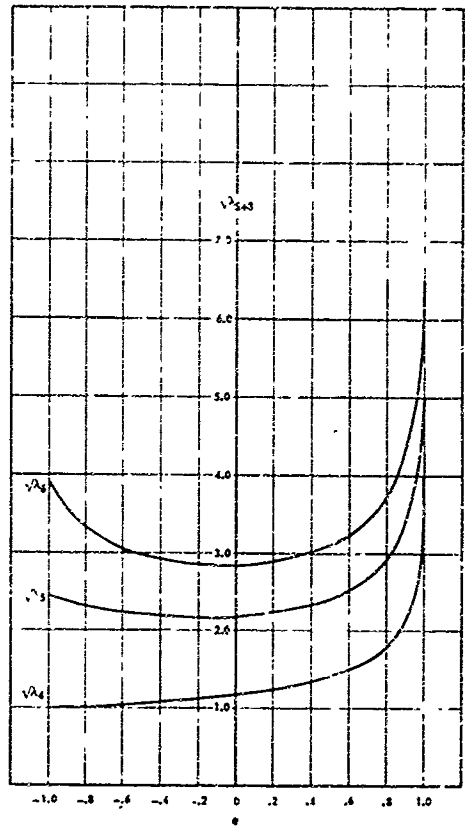


Figure 6. Variation of Cylindrical Frequency Parameter with Fuel Height Parameter

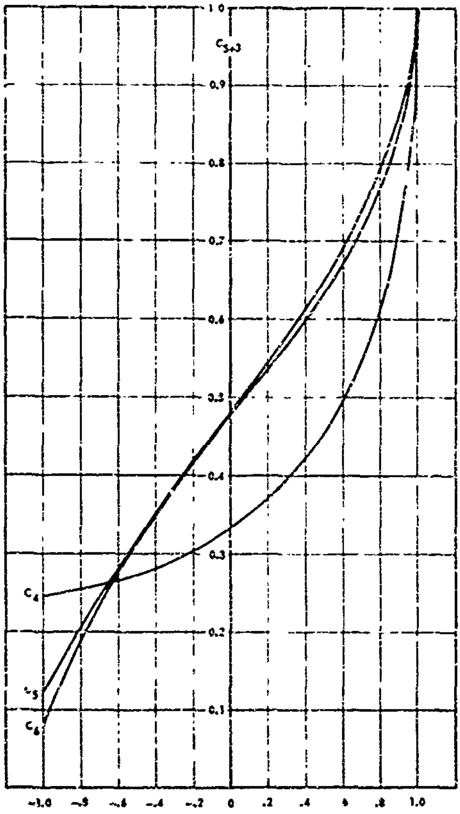


Figure 7. Variation of C_{5+3} with Fuel Height Parameter, e, Spherical Tank

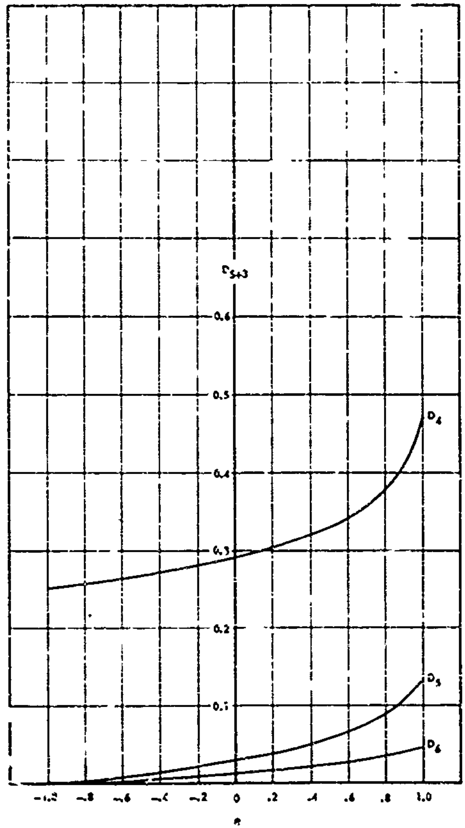


Figure 8, Variation of D_{3+3} with Fuel Height Parameter, e, - Spherical Tank

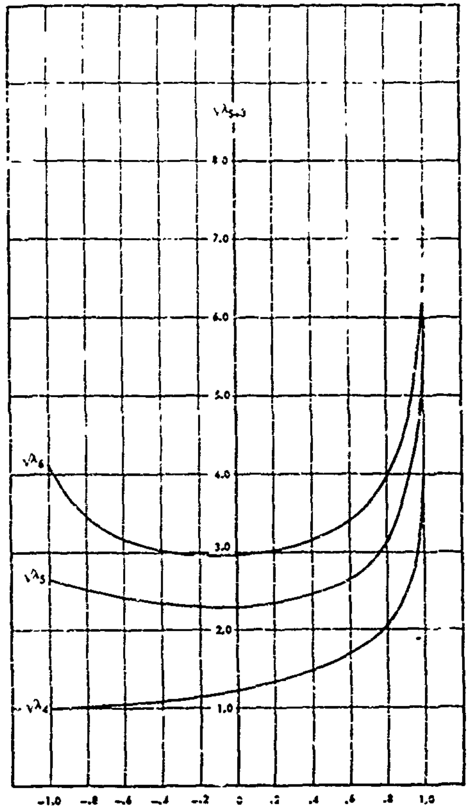


Figure 9. Variation of Spherical Frequency Parameter with Fuel Height Parameter

PART II

The computations called for in this part are carried out in the Vehicle Physical Characteristics Subprogram (VPCS2), which runs with the basic SDF program rather than the SLP.

The free surface of the fuel is parallel to the horizon only when the tank has no lateral or longitudinal acceleration. It is anticipated, however, that such will not usually be the case. The assumption now being made concerning the fuel orientation is that the free surface is always perpendicular to the "resultant tank acceleration," defined as the actual acceleration at the tank center due to the gross motion of the vehicle minus the force per unit mass due to gravity. The pertinent angles for the tank orientation, therefore, are not angles usually defined as the tank or vehicle pitch, roll, and yaw angles; they are the angles between the body axis system and the resultant acceleration. This means that at any instant of time the resultant tank acceleration must first be found and then the free surface of the fuel set perpendicular to it. Since the fuel slosh equations presented in Part I are valid for only vertical or horizontal tanks, the tank walls must be set perpendicular and parallel to the free surface. As this is done, the <u>real</u> tank dimensions in the body axis system are replaced by those of a different but "equivalent" tank of the same volume. This equivalent tank is, therefore, a tank whose dimensions and orientation are a function of the angles the real tank makes with the resultant tank acceleration. As the real tank for example pitches from 0° to 90°, the equivalent tank concept provides a continuous transition to classify the tank as being either vertical or horizontal. The tank geometrical center was chosen as being common to both the real tank and the equivalent tank.

The equivalent tank concept is by no means an exact representation but does give an approximation of the real situation. One very significant parameter to fuel sloshing is the length of the free surface. The equivalent tank concept permits the free surface length to increase or decrease as it does in the real situation, but only approximates the actual free surface length. This concept also simplifies the computation of the moments and products of inertia and the C.G. of the fuel, as the fuel changes its gross position in the tank due to the gross motion of the vehicle.

Consider now the problem of obtaining the equivalent rectangular tank dimensions and then the moments of inertia and C.G. of the fuel in the equivalent tank as if the fuel were solidified. As shown in Figure 10, λ_3 is the unit

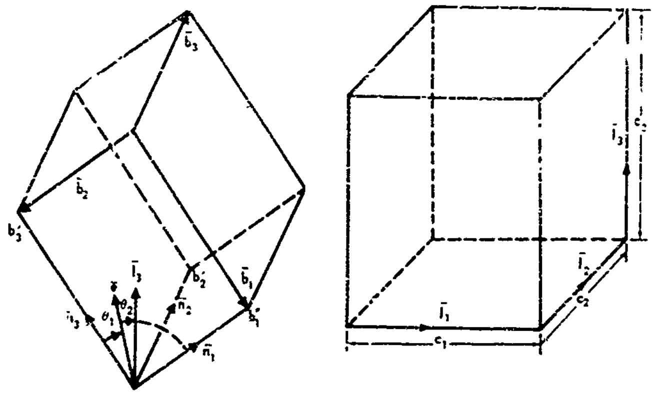


Figure 10. Real and Equivalent Rectangular Tanks

acceleration vector of the tank. One corner of the real tank is chosen as the origin of a right handed triad having unit vectors $\bar{n}_1$, $\bar{n}_2$ and $\bar{n}_3$, pointing along adjacent edges of the tank and chosen so that the angle between $\bar{l}_3$ and $\bar{n}_1$ is not less than the angle between $\bar{l}_3$ and either $\bar{n}_2$ or $\bar{n}_3$. The vertex for the common origin of these vectors is chosen so that these angles are not greater than $\pi/2$. The unit vector $\bar{e}$ is defined as a unit vector perpendicular to $\bar{n}_1$, lying in the plane of $\bar{l}_3$ and $\bar{n}_1$, and making an acute angle with $\bar{l}_3$. The vectors $\bar{L}_1$, $\bar{L}_2$ and $\bar{L}_3$ are the vectors defining the real tank size and orientation. The magnitudes of these vectors (not necessarily respectively) ℓ_1', ℓ_2' and ℓ_3' are the lengths of the sides of the tank in the direction of the unit vectors $\bar{n}_1$, $\bar{n}_2$ and $\bar{n}_3$ respectively, as shown in Figure 10. The unit vector $\bar{e}$ and the angles θ_1, and θ_2 can be defined as:

$$\bar{e} = \frac{\bar{l}_3 - (\bar{l}_3 \cdot \bar{n}_1)\bar{n}_1}{\sqrt{1-(\bar{l}_3 \cdot \bar{n}_1)^2}}$$

$$\theta_2 = \mathrm{ARC\ COS}\,(\bar{l}_3 \cdot \bar{e})$$

$$= \mathrm{ARC\ COS}\,\sqrt{1-(\bar{l}_3 \cdot \bar{n}_1)^2}$$

$$\theta_1 = \mathrm{ARC\ COS}\,(\bar{n}_3 \cdot \bar{e})$$

$$= \mathrm{ARC\ COS}\,[\bar{l}_3 \cdot \bar{n}_3 \sec\theta_2]$$

where θ_1 and θ_2 must be positive acute angles.

The dimensions of the equivalent rectangular tank can now be obtained. Referring to Figure 10, the equivalent tank may be thought of as the tank obtained by taking the real tank, with $\bar{l}_3$ coincident with one of its edges, and then adjusting the real tank dimensions to the equivalent tank dimensions as the tank is rotated first through θ_1, and then through θ_2. This then replaces the real tank, which is actually in the position described by θ_1 and θ_2 but with its sides not perpendicular to the free surface, by an equivalent tank with the same volume and approximately the same free surface length with its sides perpendicular to the free surface. Defined below are the equivalent tank dimensions C_1, C_2, and C_3 in terms of the real tank dimensions ℓ_1', ℓ_2' and ℓ_3'. The intermediate tank dimension C' is defined as the length of C_3 after the tank has been rotated through θ_1 but not through θ_2.

$$C' = \sqrt{L'_1 L'_3}\ \mathrm{TAN}\left[\theta_1 + \left(1 - \frac{4\theta}{\pi}\right) \mathrm{ARC\ TAN} \sqrt{L'_3 / L'_2}\right]$$

$$C_1 = \sqrt{L'_1 C'}\ \mathrm{COT}\left[\theta_2 + \left(1 - \frac{4\theta_2}{\pi}\right) \mathrm{ARC\ TAN} \sqrt{C'/L'_1}\right]$$

$$C_2 = \sqrt{L'_2 L'_3}\ \mathrm{COT}\left[\theta_1 + \left(1 - \frac{4\theta}{\pi}\right) \mathrm{ARC\ TAN} \sqrt{L'_3 / L'_2}\right]$$

$$C_3 = \sqrt{L'_1 C'}\ \mathrm{TAN}\left[\theta_2 + \left(1 - \frac{4\theta_2}{\pi}\right) \mathrm{ARC\ TAN} \sqrt{C'/L'_1}\right]$$

The unit vectors giving the new [illegible] are.

$$\bar{\ell}_1 = \frac{\bar{n}_1 - (\bar{\ell}_3 \cdot \bar{n}_1)\,\bar{\ell}_3}{\sqrt{1 - (\bar{\ell}_3 \cdot \bar{n}_1)^2}}$$

$$\bar{\ell}_2 = \frac{\bar{\ell}_3 \times \bar{n}_1}{\sqrt{1 - (\bar{\ell}_3 \cdot \bar{n}_1)^2}}$$

Equations for the fuel moments of inertia, J_1, J_2 and J_3, as if the fuel were solidified, taken about the fuel C.G., and the C.G. location $\bar{\bar{Z}}_F$ of the fuel, measured from the equivalent tank center along the ℓ_3 axis are shown below. The total mass of fuel in the tank is M_f and the fuel height along ℓ_3 is h.

$$J_1 = \frac{1}{12} M_F \left[(C_2)^2 + h^2\right]$$

$$J_2 = \frac{1}{12} M_F \left[(C_1)^2 + h^2\right]$$

$$J_3 = \frac{1}{12} M_F \left[(C_1)^2 + (C_2)^2\right]$$

$$\bar{\bar{Z}}_F = -\frac{1}{2}\, \bar{\ell}_3\, (C_3 - h)$$

$$h = M_F / \rho C_1 C_2$$

An approach similar to that used for the rectangular tank is presented for the cylindrical tank. There are two major differences between the equivalent rectangular and cylindrical tanks. The cross section of the equivalent tank, taken perpendicular to the resultant acceleration, is always rectangular for the equivalent rectangular tank. This cross section for the equivalent cylindrical tank may be rectangular or circular depending on the angle between the

resultant acceleration and the real tank length vector. If this cross section is circular the equivalent tank is considered as being vertical, and if rectangular, the equivalent tank is considered as being horizontal. The second difference between the equivalent rectangular and cylindrical tank concerns the angles used to specify their orientation. For the equivalent rectangular tank two angles were needed. For the equivalent cylindrical tank, the angle between the resultant acceleration and the tank length vector $\bar{l}$ is the only angle needed to specify the tank orientation.

As shown in Figure 11, $\bar{l}$ is the cylindrical tank length vector, $\bar{l}_3$ is the resultant acceleration unit vector and θ is the angle between them. Defining $\bar{k}$ as the tank length unit vector then

$$\bar{k} = \bar{l}/l$$

$$u = \bar{l}_3 \cdot \bar{k}$$

$$\theta = \text{ARCCOS } |u|$$

If $|u|$ is greater than $1/\sqrt{2}$ the equivalent tank is vertical. Defining L and R as the real tank length and radius respectively, and L_V and R_V as the equivalent tank length and radius respectively, then the relation between them is:

$$L_V = \sqrt{2LR} \quad \text{TAN}\left[\theta + \left(1 - \frac{4\theta}{\pi}\right) \text{ARC TAN} \sqrt{L/2R}\right]$$

$$R_V = \sqrt{R^2 L / L_V}$$

$$h = M_F / \pi \rho R_V^2$$

Equations for the moments of inertia of the fuel, as if the fuel were solidified, taken about the fuel C.G., J_1, J_2 and J_3, and the C.G. location $\bar{Z}_F$ of the fuel measured along $\bar{l}_3$ from the tank center are shown below.

$$J_1 = J_2 = \frac{1}{12} \pi \rho R_V^2 h (3R_V^2 + h^2)$$

$$J_3 = \frac{1}{2} \pi \rho R_V^4 h$$

$$\bar{Z}_F = -\frac{1}{2} \bar{l}_3 (L_V - h)$$

If $u^2 \neq 1.0$, the components of the unit vectors $\bar{l}_1$ and $\bar{l}_2$ giving the new directions are:

$$l_1^1 = -\frac{l_3^1 l_3^3}{\sqrt{1-(l_3^3)^2}} \qquad l_1^2 = \sqrt{1-(l_3^3)^2} \qquad l_1^3 = -\frac{l_3^2 l_3^3}{\sqrt{1-(l_3^3)^2}}$$

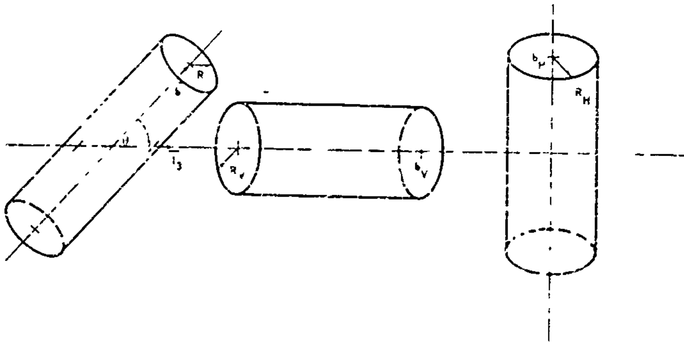


Figure 11. Resultant Acceleration and Cylindrical Tank

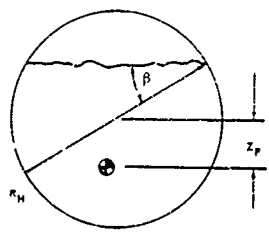


Figure 12. Horizontal Cylindrical Tank and Spherical Tank

$$\ell_2^1 = -\frac{\ell_3^3}{\sqrt{1-(\ell_3^1)^2}} \qquad \ell_2^2 = 0 \qquad \ell_2^3 = \frac{\ell_3^1}{\sqrt{1-(\ell_3^1)^2}}$$

If $u \neq \pm 1.0$, the components of $\bar{\ell}_1$ and $\bar{\ell}_2$ are

$$\ell_1^1 = \frac{k^1 - u\ell_3^1}{\sqrt{1-(u)^2}} \qquad \ell_1^2 = \frac{k^2 - u\ell_3^2}{\sqrt{1-(u)^2}} \qquad \ell_1^3 = \frac{k^3 - u\ell_3^3}{\sqrt{1-(u)^2}}$$

$$\ell_2^1 = \frac{\ell_3^2 k^3 - \ell_3^3 k^2}{\sqrt{1-(u)^2}} \qquad \ell_2^2 = \frac{\ell_3^3 k^1 - \ell_3^1 k^3}{\sqrt{1-(u)^2}} \qquad \ell_2^3 = \frac{\ell_3^1 k^2 - \ell_3^2 k^1}{\sqrt{1-(u)^2}}$$

If $|u|$ is less than or equal to $\frac{1}{\sqrt{2}}$ the equivalent cylindrical tank is horizontal. Defining L_H and R_H as the equivalent horizontal tank length and radius respectively, and expressing them as functions of the real tank dimensions L and R gives

$$L_H = \sqrt{2LR} \quad COT\left[\theta + \left(1 - \frac{2\theta}{\pi}\right) \; ARC\,TAN \sqrt{L/2R}\right]$$

$$R_H = \sqrt{R^2 L / L_H}$$

In order to obtain the expressions for the moments of inertia and C.G. of the horizontal tank, the angle β , shown in Figure 1?, must be found. The equation relating β to the fuel mass M_F is

$$M_F = \rho L_H R_H^2 \left(\frac{\pi}{2} + \beta + SIN\beta \, COS\beta\right)$$

$$\left[(M_F / \rho L_H R_H^2) - \frac{\pi}{2}\right] = \beta + \sin\beta \cos\beta$$

Now let $C_H = \left[(M_F / \rho L_H R_H^2) - \pi/2\right]$. Newton's method can then be used to find β. Defining $f(\beta)$ and $f'(\beta)$ as shown below:

$$f(\beta) = \beta + SIN\beta \, COS\beta - C_H$$

$$f'(\beta) = 2 \; COS^2\beta$$

Let β_0 be the initial estimate of β and calculate β_1

$$\beta_0 = \frac{1}{2} C_H$$

$$\beta_1 = \beta_0 - \frac{f(\beta_0)}{f'(\beta_0)}$$

The quantities $\left|\frac{f(\beta_0)}{f'(\beta_0)}\right|$ and $|f(\beta_0)|$ can then be tested to determine if they are both less than (1×10^{-7}), the arbitrarily chosen degree of accuracy. If this is true then $\beta = \beta_1$. If the desired degree of accuracy has not been obtained, β_1 is used as the next estimate of β, and a β_2 must be calculated. The test is made again to determine if the desired value of β has been obtained, and if not, the iteration must be continued until the desired conditions are satisfied. The fuel moments of inertia, C.G. and height can then be obtained as functions of β.

$$h = R_H(1+\sin\beta)$$

$$Z_F = -\frac{2R_H\cos^3\beta}{3[\pi/2+\beta+\sin\beta\cos\beta]}$$

$$J_1 = M_F\left[\tfrac{1}{2}R_H^2 + Z_F\left(\tfrac{1}{2}R_H\sin\beta - Z_F\right)\right]$$

$$J_2 = M_F\left[\tfrac{1}{4}R_H^2 + \tfrac{1}{12}\ell_H^2 - Z_F\left(Z_F - \tfrac{2}{3}R_H\sin\beta\right)\right]$$

$$J_3 = M_F\left[\tfrac{1}{4}R_H^2 + \tfrac{1}{12}\ell_H^2 - \tfrac{1}{4}R_H Z_F\sin\beta\right]$$

$$\bar{Z}_F = Z_F\,\bar{\ell}_3$$

The spherical tank dimensions do not need adjustment because for any tank orientation, the free surface length will remain unchanged. The orientation of the free surface within the tank will, however, change positions in the tank. The angle β for the spherical tank is defined in Figure 12, and the same iteration method as described previously must be used to solve for β. The following equations must be used for this iteration.

$$M_F = \pi\rho R^3\left(\tfrac{2}{3} + \sin\beta - \tfrac{1}{3}\sin^3\beta\right)$$

$$\left[\frac{M_F}{\pi\rho R^3} - \frac{2}{3}\right] = \sin\beta - \tfrac{1}{3}\sin^3\beta$$

$$C = \left[\frac{M_F}{\pi \rho R^3} - \frac{2}{3} \right]$$

$$f(\beta) = SIN\,\beta - \frac{1}{3} SIN^3\beta - C$$

$$f'(\beta) = COS^3\beta$$

$$\beta_0 = C$$

Using the value of β obtained from the iteration, the following equations for the moment of inertia, fuel height, and C.G. can be solved.

$$Z_F = -\frac{3R\,COS^4\beta}{4(2+3SIN\beta - SIN^3\beta)}$$

$$h = R(1+SIN\beta)$$

$$J_1 = J_2 = \frac{1}{5} M_F (2R^2 + 3Z_F R\,SIN\beta - 5Z_F^2)$$

$$J_3 = \frac{2}{5} M_F (R^2 - Z_F R\,SIN\beta)$$

$$\bar{Z}_F = Z_F \bar{\ell}_3$$

The components of $\bar{\ell}_1$ and $\bar{\ell}_2$ giving the new directions are:

$$\ell_1^1 = -\frac{\ell_3^1 \ell_3^2}{\sqrt{1-(\ell_3^2)^2}} \qquad \ell_1^2 = \sqrt{1-(\ell_3^2)^2} \qquad \ell_1^3 = -\frac{\ell_3^3 \ell_3^2}{\sqrt{1-(\ell_3^2)^2}}$$

$$\ell_2^1 = -\frac{\ell_3^3}{\sqrt{1-(\ell_3^2)^2}} \qquad \ell_2^2 = 0 \qquad \ell_2^3 = \frac{\ell_3^1}{\sqrt{1-(\ell_3^2)^2}}$$

PART III

Inasmuch as the length of an "equivalent" vertical cylindrical tank is most likely to be different from that of the real tank, and since the bending mode shape $f(x)$ is given for the length of the real tank but must be applied along the length of the equivalent tank, it is necessary to find some way of adapting the use of $f(x)$ to a changing tank length.

To accomplish this, a new argument x_i' is introduced which does not vary with the tank length, and f is given as $f(x_i')$. Separate consideration must be given to the two cases $u_i > 1/\sqrt{2}$ and $u_i < -1/\sqrt{2}$, where u_i equals $\cos\theta_i$ and for a vertical "equivalent" tank $|u_i| > 1/\sqrt{2}$. x_i' is defined as distance along the axis measured from the center of cylindrical tank i, nondimensionalized with respect to the tank length. In use with the real tank, x_i' is positive in the direction so chosen in connection with the submission of data to the VPC82. In use with a vertical equivalent tank, x_i' is positive in the direction of the "resultant acceleration" if $u_i > 1/\sqrt{2}$, and positive in the direction opposite to the "resultant acceleration" if $u_i < -1/\sqrt{2}$.

For the submission of data, we note that

l_i is the length of the real tank, that $l_i x_i'$ is actual distance measured in feet from the center of the real tank, that $-\frac{1}{2} \le x_i' \le \frac{1}{2}$, and that

$$f_i' = \frac{df_i}{dx_i} = \frac{df_i}{dx_i'}\frac{dx_i'}{dx_i} = \frac{df_i}{dx_i'}/l_i .$$

We next consider the mathematical relations connected with the equivalent tank.

When $u_i > 1/\sqrt{2}$, these are as shown below:

$$x_i = l_{vi}\, x_i' + (l_{vi} - h_i)/2$$

$$\frac{x_i}{h_i} = \frac{l_{vi}}{h_i}\left(\frac{1}{2} + x_i'\right) - \frac{1}{2}$$

$$dx_i = l_{vi}\, dx_i'$$

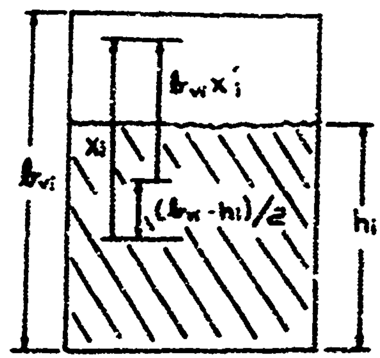
$$x_i' = \frac{x_i - (l_{vi} - h_i)/2}{l_{vi}}$$

$$= \frac{x_i}{l_{vi}} - \left(1 - \frac{h_i}{l_{vi}}\right)/2$$

$$= \frac{h_i}{l_{vi}}\left(\frac{1}{2} + \frac{x_i}{h_i}\right) - \frac{1}{2}$$

When $x_i = -\frac{h_i}{2}$, $x_i' = -\frac{1}{2}$

When $x_i = \frac{h_i}{2}$, $x_i' = \frac{h_i}{l_{vi}} - \frac{1}{2}$.



"Equivalent Tank"

$$F'_{oi} = \frac{b_{vi}}{h_i}\int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}}-\frac{1}{2}} f_i \, dx'_i$$

$$F'_{si} = \frac{b_{vi}}{h_i}\int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}}-\frac{1}{2}} f_i \cos\left[2\pi \frac{b_{vi}}{h_i}\left(\frac{1}{2} + x'_i\right)\right] dx'_i$$

$$f'_{wi} = f'_i\left(-\frac{1}{2}\right)$$

$$f'_{ti} = f'_i\left(\frac{h_i}{b_{vi}} - \frac{1}{2}\right)$$

$$G_i = \frac{b_{vi}}{h_i}\int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}}-\frac{1}{2}} x'_i f_i \, dx'_i$$

When $u_i < -\frac{1}{\sqrt{2}}$, the pertinent relations are as follows:

$$X_i = (b_{vi} - h_i)/2 - b_{vi} X'_i$$

$$dX_i = -b_{vi}\, dX'_i$$

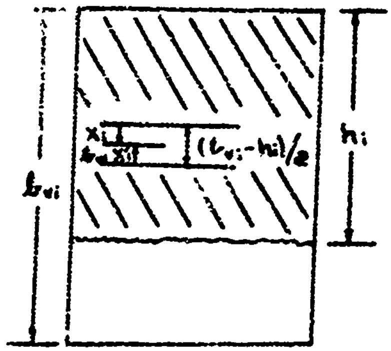
$$\frac{X_i}{h_i} = \frac{b_{vi}}{h_i}\left(\frac{1}{2} - X'_i\right) - \frac{1}{2}$$

$$X'_i = \frac{(b_{vi} - h_i)/2 - X_i}{b_{vi}}$$

$$= \frac{1}{2} - \frac{h_i}{b_{vi}}\left(\frac{1}{2} + \frac{X_i}{h_i}\right)$$

When $X_i = -\frac{h_i}{2}$, $X'_i = \frac{1}{2}$

When $X_i = \frac{h_i}{2}$, $X'_i = \frac{1}{2} - \frac{h_i}{b_{vi}}$



"Equivalent Tank"

$$F'_{oi} = -\frac{b_{vi}}{h_i}\int_{\frac{1}{2}}^{\frac{1}{2}-\frac{h_i}{b_{vi}}} f_i \, dX'_i = \frac{b_{vi}}{h_i}\int_{\frac{1}{2}-\frac{h_i}{b_{vi}}}^{\frac{1}{2}} f_i \, dx'_i$$

$$F'_{si} = -2\,\frac{\ell_{vi}}{h_i}\int_{\frac{1}{2}}^{\frac{1}{2}-\frac{h_i}{\ell_{vi}}} f_i \cos\left[s\pi\,\frac{\ell_{vi}}{h_i}\left(\tfrac{1}{2}-x'_i\right)\right] d\,x'_i$$

$$= 2\,\frac{\ell_{vi}}{h_i}\int_{\frac{1}{2}-\frac{h_i}{\ell_{vi}}}^{\frac{1}{2}} f_i \cos\left[s\pi\,\frac{\ell_{vi}}{h_i}\left(\tfrac{1}{2}-x'_i\right)\right] d\,x'_i$$

$$f'_{bi} = f'_i\left(\tfrac{1}{2}\right)$$

$$f'_{ti} = f'_i\left(\tfrac{1}{2}-\frac{h_i}{\ell_{vi}}\right)$$

$$G_i = -\,\frac{\ell_{vi}}{h_i}\int_{\frac{1}{2}-h_i/\ell_{vi}}^{\frac{1}{2}} x'_i f_i \, d\,x'_i$$

PART IV

As indicated in equations (97) and (98), provisions have been made to include damping in the SLP. Knowledge of what numerical values to use for ϕ'_j (or δ_j) is important to a careful investigation of fuel sloshing in a vehicle in flight. An extensive literature search found that very little fuel damping data exists except for upright cylindrical tanks. Reference (6) did present the equation below, which can be used to obtain the logarithmic decrement δ_j for an upright cylindrical tank as a function of the kinematic viscosity ν, the fuel height h, the acceleration due to gravity g, and the tank radius R:

$$\delta = \frac{5.23\sqrt{\nu}\left[1+2(1-h/R)\,\mathrm{csch}\,(3.68h/R)\right]}{\left[R^3 g \tanh\,(1.84h/R)\right]^{1/4}}$$

This equation is for a tank with no baffles. Most of the other references found were for upright cylindrical tanks with various baffling configurations.

Because of the scarcity of data on fuel damping, no equations such as the one just given (which is of limited applicability) are employed in the SLP. Rather, it is left to the user to determine in his own way constant values of ϕ'_j for submittal as input to the program. As long as the submitted values of ϕ'_j are greater than zero, they will at least prevent the infinite continuation of whatever fuel slosh modes are excited by the motion of the vehicle.

APPENDIX II

SYMBOLS, DATA TO BE SUBMITTED, COMPUTATIONS AND EQUATIONS USED IN THE STRUCTURAL LOADS PROGRAM

Symbols

A^{-} components in the y coordinate system of the linear acceleration of the vehicle at the origin of the vehicle axes. SIAR7T

$A^{r}_{Ei h}$ components in the y coordinate system of the linear acceleration of the h-th particle of the i-th section due to elastic deformation. ADAEHT

$A^{r}_{Ri h}$ components in the y coordinate system of the linear acceleration of the h-th particle of the i-th section due to rigid motion. ADARHT

A^{rs}_{j} static aerodynamic terms.

A_{si} nondimensional fuel slosh modal parameters for lateral motion of horizontal cylindrical tanks.

A'_{si} a quantity used with vertical cylindrical tanks. TAAPS

$A^{\prime rsb}_{i}$ sectional aerodynamic shear force terms for rigid vehicle, referred to vehicle axes. SAAPIT

$A^{\prime\prime rsb}_{i}$ sectional aerodynamic bending moment terms for rigid vehicle, referred to vehicle axes. SAAPPT

a^{r}_{ih} $A^{r}_{Eih} + A^{r}_{Rih}$ components in the y coordinate system of the linear acceleration of the h-th particle of the i-th section. ADAIHT

a'^{rs}_{ji} inertia terms.
SFLAPS

$a'^{r}_{ki\lambda}$ components in the ν'_i coordinate system of the given mode of vibration in degree of freedom k before balancing.

a'^{r}_{ki} components in the ν'_i system of the given partial linear velocity with respect to q^k of the point of rotation of movable section i relative to the vehicle before balancing.

$a\lambda_{si}$ modal functions of cylindrical tank aspect ratios.
TAAR

$a\lambda_{usi}$ modal functions of rectangular tank aspect ratios.
TAAR

B_{si} nondimensional fuel slosh modal parameters for lateral motion of horizontal cylindrical tanks.

B^{rs}_{jk} aerodynamic stiffness terms.

B'^{rst}_{ki} sectional aerodynamic shear force terms, referred to vehicle axes.
SABPIT

B''^{rst}_{ki} sectional aerodynamic bending moment terms, referred to vehicle axes.
SABPPT

b_{Hi} lengths of equivalent horizontal cylindrical tanks. Same as in VPCS.
TABHTT

ℓ_{vi} lengths of equivalent vertical cylindrical tanks. Same as in VPCS.
TALV7T

β^r_k components of the dynamically balancing rotation rate with respect to q^* of the vehicle relative to the vehicle axes.
SEBF7T - STRUCTURE
TABF7T - TANKS

C_{si} nondimensional fuel slosh modal parameters for lateral motion of spherical tanks.

C^r_{jk} aerodynamic damping terms.

C'^{rs}_{ki} sectional aerodynamic shear force terms referred to vehicle axes.
SACPIT

C''^{rs}_{ki} sectional aerodynamic bending moment terms referred to vehicle axes.
SACPPT

c_{1i} , c_{2i} lengths of the "horizontal" edges of the (equivalent) rectangular tanks. Same as in VPCS.
TACS7T

c^r_k components of the dynamically balancing translation rate with respect to q^* of the vehicle relative to the vehicle axes.
SECTKT - STRUCTURE
TACTKT - TANKS

D_{si} nondimensional fuel slosh modal parameters for lateral motion of spherical tanks.

D_{rs} moments and negatives of products of inertia of structure and fuel about axes thru the vehicle center of mass and parallel to the vehicle axes.
TA-77S

D^r_{ih} deflections of the h-th particle of the i-th section due to elastic deformation.
ADDIHT

$\mathfrak{D}_{rsk}$ modal products of inertia of part of the vehicle.
SDDKS

d^r_{jk} dynamic balancing term.
SFDJK

E number of thrust vectoring nozzles (or "engines").
NØENG

E'_k modal inertia term.
SFEK7S

e base of natural log system.

e^r_{si} components in the $\bar{J}_r$ system of the $\bar{J}'_{si}$ vector. Same as in VPCS.
TAESRT

F_{rs} moments and negatives of products of inertia of fuel about vehicle axes.
TAFR7S

F'_{si} certain integrals connected with fuel slosh in vertical cylindrical tanks.
TAFPS

f_{k} bending mode shapes for cylindrical tanks.
FTABLE

f'_{ki} slopes of bending mode shapes for cylindrical tanks.
FPTABL

f'_{kbi} f'_{ki} at the bottom of the fluid in tank i
TAFPB

f'_{ksi} f'_{ki} at the top (or surface) of the fluid in tank i.
TAFPT

G_{rs} products of inertia of structure and fuel about vehicle axes.
SFGRSS

$\mathfrak{G}_{rs}$ products of inertia for part of the vehicle.
SIGLRS

G_i, G'_i, G''_i, G'''_i integrals connected with fuel slosh in vertical cylindrical tanks.
TAGI, TAGPI, TAGPP, TAG3P

g_i the magnitude of the "resultant acceleration" at the center of tank i. Same as in VPCS.
TAGI7T

g^r_a components of force per unit mass due to gravity. Same as in VPCS.
SGGRAP

g'_j the coefficient of "structural" damping associated with the j-th degree of freedom.
AEGPJ

H_{jK} inertia coupling terms.

H_k^r modal unbalances.
SFHTKS

h_i height of fuel in "equivalent" tank i. Same as in VPCS.
TAHAIT

h_{ji}^r components of the partial linear velocity with respect to q^r of the center of mass of section i relative to the vehicle axes -- values obtained after dynamic balancing.
TAHKI
SFHKI

$\underset{\smile}{HF}_k^r$ modal unbalances for fuel.
TAHFTS

$\underset{\smile}{HF}_{jk}$ inertia coupling terms for fuel.
TAHFJS

$\underset{\smile}{HS}_{jk}$ inertia coupling terms for structure.
SEHSJS

$\underset{\smile}{HS}_k^r$ modal unbalances for structure.
SEHSTS

H'_{sti} moments and negatives of products of inertia for part of a section.
SIHHIS

I_{rs} moments and negatives of products of inertia of structure and fuel about vehicle axes.
SFIRSS

I_{Fri} moments of inertia of fuel in "equivalent" tanks about axes parallel to vehicle axes.

$\mathbb{I}_{rs}$ moments and negatives of products of inertia for part of the vehicle.
SIIIRS

J_{ri} moments of inertia of section i about sectional axes. Same as in VPCS.
TAJR7S

J_{Fri} moments of inertia of fuel as if it were solid in "equivalent" tanks about tank axes. Same as in VPCS.
TAJF7S

J'_{Fri} effective moments of inertia of fuel about tank axes.
TAJFPS

γ^{r}_{ki} components of the partial linear velocity with respect to q^{k} of the center of mass of section i relative to the vehicle axes -- arbitrary values given prior to dynamic balancing.
TAJTI - TANKS
SEJKI - STRUCTURE

K_{ri} products of inertia of section i referred to sectional axes. Same as in VPCS.
TAKR7S

L^{rs}_{k} modal inertia terms.
SFLSTS

ℓ^{r}_{ui} components of orthogonal unit_vectors giving directions of acceleration oriented axes, ℓ_{1i} and ℓ_{2i} being parallel to the surface of the fuel in tank i, and ℓ_{3i} being perpendicular to the surface of the fuel. Same as in VPCS.
TALSRT

$\underset{\smile}{LF}{}^{st}_{k}$ modal inertia terms for fuel.
TALFTS

$\underset{\smile}{LS}{}^{st}_{k}$ modal inertia terms for structure.
SELSTS

M_{Fi} total masses of fuel in tanks. Same as in VPCS.
TAMF7S

M_{JK} components of the inertia tensor.
SPMJKS

M^r bending moments at a specified location on flexible vehicle without wind (components in the y [illegible] system).
SPM77T

M^r_b bending moments on flexible vehicle [illegible]
SPMF7T

M^r_{R1} bending moments on rigid vehicle without wind and without thrust forces.
SPMR1T

M^r_{R2} bending moments on rigid vehicle without wind but with thrust forces.
SPMR2T

M^r_{Rb} bending moments on rigid vehicle with wind and with thrust forces.
SPMRBT

$\underset{\smile}{MA}{}^r_{E\sigma}$ aerodynamic bending moments about the origin due to elastic deformation, without wind.
SAMAET

$\underset{\smile}{MA}{}^r_{R\sigma}$ aerodynamic bending moments about the origin due to rigid motion, without wind.
SAMART

$\underset{\smile}{MA}{}^r_{Eb\sigma}$ aerodynamic bending moments about the origin due to elastic deformation, with wind.
SAMEBT

$\underset{\smile}{MA}{}^r_{Rb\sigma}$ aerodynamic bending moments about the origin due to rigid motion, with wind.
SAMRBT

$\underset{\sim}{MG}{}^{r}_{o}$ bending moments about the origin due to gravity.
SGMGRT

$\underset{\sim}{MI}_{E\theta}$ inertial bending moments about the origin due to elastic deformation.
SIMIET

$\underset{\sim}{MI}{}^{r}_{R\theta}$ inertial bending moments about the origin due to rigid motion.
SIMIRT

$\underset{\sim}{MT}{}^{r}_{R\theta}$ bending moments about the origin due to the thrust forces of the engines.
STMTRT

m total mass of vehicle and fuel at any instant. Same as in VPCS.
AMASS

m_i mass of structural section i. Same as in VPCS.
TAMI7S

m_{ih} mass of the h-th particle of section i.
SEMIH

m_{ki} effective fuel slosh masses in tank i.

m' mass of part of the vehicle and fuel.
SIMP7S

m_i' mass of part of section i.
SIMPIS

m'_{ki} fuel slosh inertia coupling terms for spherical and horizontal cylindrical tanks.
TAMPKS

N_i number of aerodynamic parts (or surfaces) in section i.
NØNI

N^r_j modal inertia term.

N^r_{Ei} sectional aerodynamic bending moment terms resulting from elastic deformation, without wind.
SANEIP

N^r_{Ri} sectional aerodynamic bending moment terms resulting from rigid motion, without wind.
SANRIP

$N^r_{E\ell i}$ sectional aerodynamic bending moment terms resulting from elastic deformation, with wind.
SANEBP

$N^r_{R\ell i}$ sectional aerodynamic bending moment terms resulting from rigid motion, with wind.
SANRBP

$\underset{\smile}{N}F^r_k$ modal inertia terms for fuel.
TAHFKS

$\underset{\smile}{N}S^r_k$ modal inertia terms for structure.
SENSKS

n number of elastic degrees of freedom.
NØEF.

n'^{r}_{ih} components in the ν' coordinate system of a unit vector at point h on the surface of section i, perpendicular to the surface and pointing outward.
AENPR

P_i number of particles (or masses) in section i
NOPI

P_{Fri} products of inertia of fuel in tank i referred to axes parallel to vehicle axes.

P_{rsj} modal moments and negatives of products of inertia of vehicle and fuel.
SEPJ7S - STRUCTURE
TAPJ7S - TANKS

p'^{r}_{i} sectional coordinates of the point of rotation of movable section i. Same as in VPCS.
SEPPR

p'^{rs}_{ji} modal moments and negatives of products of inertia of section i.

$\mathcal{P}_{rsk}$ modal moments and negatives of products of inertia of part of the vehicle.
SIPPKS

$\textcircled{T}_j$ the generalized forces associated with thrust forces.
EMQTJT

q^j generalized coordinate associated with the j-th degree of freedom.
EMQ7T

q'^{rs}_{ki} modal moments and negatives of products of inertia of part of section i.
SIPPKS

R_i radius of spherical tank i. Same as in VPCS.
TARIPT

R_{Hi} radius of equivalent horizontal cylindrical tank i. Same as in VPCS.
TARH7T

R_{Vi} radius of equivalent vertical cylindrical tank i. Same as in VPCS.
TARV7T

R^r_k modal products of inertia of part of the vehicle referred to vehicle axes.
SIRKRS

R^{rs}_{ji} sectional aerodynamic force terms for rigid vehicle.
AERTUT

R'^{rst}_i sectional aerodynamic shear force terms for rigid vehicle.
SARPIT

R''^{rstu}_i sectional aerodynamic bending moment terms for rigid vehicle.
SARP1T, SARP2T, SARP3T

$\mathbb{R}^r_k$ a term of R^r_k.
SIRRKS

r_{1i}, r_{2i} aspect ratios of "equivalent" tanks. Same as in VPCS.
TARTL

r_{3i} a quotient of aspect ratios of rectangular tank i.
TARTL

S [illegible] of [illegible] in [illegible].
NØSN

S_{rs} moments and negative [illegible] products of inertia of structure of entire vehicle about [illegible] in [illegible].
TASR73

S_{ik} area of the k-th [illegible] of the i-th section.
AWSIKT

S^r shear forces at a specified location on flexible vehicle without wind (components in the r coordinate system).
SPST7P

S^r_b shear forces on flexible vehicle with wind.
SPSB7P

S^r_E shear forces due to elastic deformation without wind.
SPSE7P

S^r_{Eb} shear forces due to elastic deformation with wind.
SPSEBP

S^r_{R1} shear forces on rigid vehicle without wind and without thrust force.
SPSR1P

S^r_{R2} shear forces on rigid vehicle without wind but with thrust forces.
SPSR2P

S^r_{Rb} shear forces on rigid vehicle with wind.
SPSRBP

S^{rs}_{jkl} sectional aerodynamic stiffness terms.
AESTUT

SA^r_E aerodynamic shear forces due to elastic deformation, without wind.
SASEAP

SA^r_R aerodynamic shear forces due to rigid motion, without wind.
SASRAP

$SA^r_{E\ell}$ aerodynamic shear forces due to elastic deformation, with wind.
SASREP

$SA^r_{R\ell}$ aerodynamic shear forces due to rigid motion, with wind.
SASRBP

SA^r_{Ei} elastic contribution of section i to the aerodynamic shear forces, without wind.
SASAEP

SA^r_{Ri} rigid contribution of section i to the aerodynamic shear forces, without wind.
SASARP

$SA^r_{E\ell i}$ elastic contribution of section i to the aerodynamic shear forces, with wind.
SASEBP

$SA^r_{R\ell i}$ rigid contribution of section i to the aerodynamic shear forces, with wind.
SASRBP

SG^r shear forces due to gravity.
SGSGRP

SI_E^r inertial shear forces due to elastic deformation.
SISIEP

SI_R^r inertial shear forces due to rigid motion.
SISIRP

ST_R^r shear forces due to the thrust forces of the engines.
STSTRP

T number of tanks. Same as in VPCS.
NØTAN

T_{xi} components in the y coordinate system of the thrust force at the i-th nozzle.
EMTXZP(1)

T_{yi} components in the y coordinate system of the thrust force at the i-th nozzle.
EMTXZP(2)

T_{zi} components in the y coordinate system of the thrust force at the i-th nozzle.
EMTXZP(3)

T_{jKi}^r sectional aerodynamic damping terms.
ASTIJT

$T_{Ki}^{\prime rs}$ sectional aerodynamic shear force terms.
SATPIT

$T_{Ki}^{\prime\prime rst}$ sectional aerodynamic bending moment terms.
SATPPT

t time.

U^{rs}_{jk} aerodynamic terms.
AUTRT

U'^{rst}_{ki} sectional aerodynamic shear force terms.
SAUPIT

U''^{rstu}_{ki} sectional aerodynamic bending moment terms.
SAUP1T, SAUP2T, SAUP3T

u_i cosine of angle between resultant acceleration and axis of cylindrical tank i. Same as in VPCS.
TAUS

V^r components in the y coordinate system of the linear velocity of the vehicle at the origin of the vehicle axes.
AEVR7T

V^r_a components of the velocity of the wind.
AEVRAT

V^r_b $V^r - V^r_a$.
AEVRBT

V^r_c components of the vehicle velocity at the center of mass.
AEVRCT

$\dot{V}^r_c$ dV^r_c/dt
SIVDCT

$v^r_{\epsilon i h}$ components of the velocity of particle h of section i relative to the vehicle axes.
ADVEHT

W^{r}_{KL} inertia coupling terms for part of the vehicle referred to vehicle axes.
SI KLS

$\mathcal{W}^{r}_{KL}$ one term of W^{r}_{KL}.

$\mathcal{W}_{i}$ number of fuel slosh modes in each direction for tank i. (≤ 2)
NØWI

$\mathcal{W}_{i h}$ the "piston speed" (or downwash) at the h-th surface of the i-th section.
AZWIHT

$\mathcal{W}^{r}_{\varepsilon i h}$ one term of $A^{r}_{\varepsilon i h}$.
ADWEHT

X'_{i} distance along the axis measured from the center of cylindrical tank i, nondimensionalized with respect to the tank length. In use with the real tank, X'_{i} is positive in the direction so chosen in the VPCS data to be submitted, number 5. In use with a vertical equivalent tank, X'_{i} is positive in the direction of the "resultant acceleration" if u_{i} is positive, and positive in the direction opposite to the "resultant acceleration" if u_{i} is negative. ($u_{i} = \cos\theta_{i}$ and in the case of a vertical "equivalent" tank $|u_{i}| > 1/\sqrt{3}$.)

X^{r}_{i} coordinates of geometric center of tank i or of the point of rotation of movable section i. Same as in VPCS.

X^{r}_{Ki} $\sum_{s=1}^{3} e^{r}_{si} X'^{s}_{Ki}$.
TAXRK - TANKS
SEXRK - STRUCTURE

χ'^{r}_{ki} $\quad \bar{a}'^{r}_{ki} - \rho'^{r}_{ki} = \gamma'^{r}_{ki} + \sum_{s}^{3} \sum_{t}^{3} C_{rst} B'^{s}_{ki} \gamma'^{t}_{i}$.

$\chi^{r}_{Ei\hbar}$ one term of $A^{r}_{Ei\hbar}$.
ADXEHT

Y^{r} static unbalances of part of the vehicle referred to vehicle axes.
SIYR7S

Y^{r}_{i} static unbalances of part of section i , referred to vehicle axes.
SIYRIS

Y^{r}_{K} dynamic unbalances of part of the vehicle referred to vehicle axes.
SIYRKS

Y^{r}_{KL} inertia coupling terms for part of the vehicle referred to vehicle axes.
SIYKLS

Y'_{si} , Y''_{si} certain summations connected with fuel slosh in vertical cylindrical tanks.
TAYPI, TAYPP

$y^{r}_{i\hbar}$ coordinates of the $\hbar$-th particle of the i-th section in the y coordinate system.
ADYIHT, SAYRET

$\mathcal{Y}^{rs}_{i}$ products of inertia of part of section i , referred partly to sectional axes and partly to vehicle axes.
SIYYIS

$\mathcal{Y}^{rs}_{Ki}$ modal products of inertia of part of section i , referred partly to sectional axes and partly to vehicle axes.
SIYBKS

Z_{ki} Z_{usi} distance in tank i from fuel center of mass to spring mass, S, positive up.
TAZIT

$\bar{Z}_i$ distance from geometric center to center of fuel mass, positive upward, for equivalent tank i. Same as in VPCS
TAZBAR

$\mathfrak{z}_i^r$ coordinates of the center of mass of section i. Same as in VPCS.
SEZSI

$\mathfrak{z}_{Fi}^r$ coordinates of center of mass of fuel in tank i. Same as in VPCS.
TAZFRT

$\mathfrak{z}_c^r$ coordinates of the centerof mass of the vehicle. Same as in VPCS.
ZCFRT

$\omega_{ji}^{\prime r}$ components of partial angular velocity with respect to q^j of the $\bar{J}_{ri}$ coordinates relative to the $\bar{J}_r$ system - values obtained after dynamic balancing.
SFAPR
TAAPR

β_i fuel height angle for spherical and horizontal cylindrical tanks; that is, the angle between the free surface and a line from the center of the tank to the intersection of the free surface with the wall of the tank. Same as in VPCS.
TABETR

β_{ki}^r components in the $\bar{J}_r$ system of the partial angular velocity with respect to q^k of the $\bar{J}_{ri}$ coordinates relative to the $\bar{J}_r$ system -- arbitrary values given prior to dynamic balancing.
SEBKI - STRUCTURE
TABKI - TANKS

$B_{ji}^{\prime r}$ components in the $\bar{J}_r'$ system of the partial angular velocity with respect to q^j of the $\bar{J}'_{ri}$ coordinates relative to the $\bar{J}_r$ system -- arbitrary values given prior to dynamic balancing.
TABRK - TANKS
SEBRK - STRUCTURES

Γ_{rsi} products of inertia of section i, referred to the sectional axes.
TACGFS - TANKS
SECGRS - STRUCTURE

Γ'_{rsi} products of inertia of part of section i, referred to the sectional axes.
SIGGIS

$\gamma_1, \gamma_2, \gamma_3$ constants obtained from Bessel functions and used with vertical cylindrical tanks.
TAGAM

Δ_{jK} inertia coupling terms.

ΔF_{jK} inertia coupling terms for fuel.
TADFJS

ΔS_{jK} inertia coupling terms for structure.
SEDGJS

Δt time increment used in the numerical integration.

ϵ_j^r modal inertia terms.
SFEPSS

$\zeta_{jKi}^{\prime rs}$ inertia coupling terms.
SIZETS

η_{jK} inertia coupling terms.

ηF_{JK}	inertia coupling terms for fuel. TATEFS
ηS_{JK}	inertia coupling terms for struct .. SETASS
θF_{J}^{rs}	modal inertia terms for fuel. TATFJS
θS_{J}^{rs}	modal inertia terms for structure. SETSJS
Λ_{K}^{r}	modal inertia terms.
Λ'^{rs}_{Ki}	modal products of inertia of section i. TACLAM - TANKS SECLAM - STRUCTURE
$\hat{\Lambda}'^{rs}_{Ki}$	modal products of inertia of part of section i. SICLLS
ΛF_{K}^{r}	modal inertia terms for fuel. TACLFS
ΛS_{K}^{r}	modal inertia terms for structure. SECLSS
λ_{si}	frequency parameters for lateral motion of horizontal cylindrical tanks.
λ'_{si}	frequency parameters for lateral motion of spherical tanks.

μ_{JK} inertia coupling terms for fuel and structure.
TAMUJS - TANKS
SEMUJS - STRUCTURE

μ_{JK} fuel slosh inertia terms for vertical cylindrical tanks.
TAU1PS, TAU2PS, TAU3PS, TAU4PS, TAUJ7S

ξ_{jih} aerodynamic modal term.
AEXIJT

π ratio of circumference to diameter of a circle.
PI

ρ the atmospheric density.
EMRHØS

ρ_i density of fuel in tank i Used with spherical and horizontal cylindrical tanks. Same as in VPC.
TARHØS

ρ'^{r}_{Ki} components in the ν_i system of the given partial linear velocity with respect to q^K of the point of rotation of movable section i relative to the section.

$\sum F_j^{rs}$ modal inertia terms for fuel.
TACSFS

$\sum S_j^{rs}$ modal inertia terms for structure.
SECSSS

σ'^{r}_{Kih} $\partial \nu'^{r}_{ih} / \partial q^K$
SESPSH, AESIPT

τ'^{r}_{jih} $\partial n'^{r}_{ih} / \partial q^j$

AETAPT

$\Upsilon_i^{\prime r}$ static unbalance of part of a section, referred to sectional axes.
SIGAPS

$\upsilon_{h}^{\prime r}$ coordinates in the $\bar{J}_{ri}^{\prime}$ system of particle h.
SEVPRH, AEVPRT, SIVEGT

Φ_{JK} fuel slosh inertia terms.
SEPJKS STRUCTURE
TAPJKS - TANKS

Φ^r_{KLih} kinematic modal coupling term such that
$\Phi^r_{KLih} + \Phi^r_{LKih} = \partial T^r_{Kih} / \partial q^L$.
ADPKHT

Ψ_{Si} a function used in connection with vertical cylindrical tanks.
TAPSI

$\Psi_{Ki}^{\prime r}$ dynamic unbalance of part of a section, referred to sectional axes.
SIPSIS

Ω^r components in the y coordinate system of the angular velocity of the vehicle axes.
ADØMDR

$\dot{\Omega}^r$ components in the y coordinate system of the angular acceleration of the vehicle axes.
SIØMDR

ω_{Ki} fuel slosh frequency in the k-th degree of freedom.
TAWKI

ω_j vibration frequency associated with the j-th degree of freedom.
SEWJ

$\urcorner^{r}_{k,h}$ $\partial \gamma^{r}_{ih} / \partial q^{k}$

ADCKET

$\boxed{Ki\ ,j}$ inertia "symbols."

SP3 75

Data to be Submitted

1. n (= number of elastic degrees of freedom).

2. For all tanks, w_i $(0 \le w_i \le 2)$; χ^r_{ki}, β^r_{ki}, for $k > 40$, that is, for the fuel k's.

3. For all cylindrical tanks,
$$\left.\begin{array}{l} f_{ki} \text{ versus } X'_i \\ f'_{ki} \text{ versus } X'_i \end{array}\right\} \left(-\tfrac{1}{2} \le X'_i \le \tfrac{1}{2}\right) \text{for } k > 40$$

4. For structure, including tanks but not fuel, ρ'^r_i, P_i

   a. If $P_i = 1$, then χ^r_{ki} and β^r_{ki} are given. (m_{ih}, v'^r_{ih}, σ'^r_{kih} are not given).

   b. If $P_i > 1$, then m_{ih}, v'^r_{ih} and σ'^r_{kih} are given ($h = 1, 2, \ldots P_i$; $r = 1, 2, 3$; $K = 41, 42, \ldots$).
   If $\sigma'^r_{kih} = 0$, then χ^r_{ki} and β^r_{ki} are given.
   If $\sigma'^r_{kih} \neq 0$, then γ'^r_{ki} and β'^r_{ki} are given.

5. For aerodynamic parts of structural sections, n'^r_{ih}, σ'^r_{jih}, v'^r_{ih}, S_{ih}, T'^r_{jih}, N_i (≤ 10)

6. For all degrees of freedom, γ'_j.

7. For structural vibration modes, ω_j.

8. For the computation of structural loads, designations of points in the structure, and, with each point, associated sections, tanks, and engines (that is, thrust vectoring nozzles). Points are designated by giving the numbers i of sections and the sectional coordinates v_{ip} of the points. With each section, there must also be an indication of which particles and which aerodynamic parts will be included in the summations. For some sections, all particles and aerodynamic parts will be used; for such sections, the user should so indicate, because this results in simplification of some formulas. Submit values of γ.

9. For the computation of accelerations and deflections, designation of points in the vehicle. Points are designated by giving the numbers i of sections or tanks and h of particles within sections for which $P_i \neq 1$. If i designates a tank or a section for which $P_i = 1$, h is not given.

Data Required from the Basic SDF Program

1. M_{r_i} for all tanks.
2. ρ, V_c^r; $V_{[illegible]}^r$ Ω^r, $\dot{\Omega}^r$, $\dot{V}_{[illegible]}^r$, $(r = 1, 2, 3)$
3. E (= number of engines. Same as in VPCS.)
4. T_{x_i}, T_{y_i}, T_{z_i}; $(i = 1, 2, \ldots E)$
   These are functions of time.
5. g_a^r

List Required from the VPC Subprogram

1. For structure, including tanks but not fuel,
   S_{rs}
   $m_i, \bar{z}_i, e^r_{si}, J_{ri}, K_{ri}$ $(r, s = 1,2,3,\ i = 1,2,\ldots 8)$
2. For the entire vehicle (structure, tanks, and fuel),
   $m, \bar{z}, S, T$
3. γ_i
4. ℓ_{ui} $(u = 1,2,3)$
5. For all tanks,
   $\gamma_{Fi}, \bar{Z}_i$
6. Information as to whether or not each equivalent tank is rectangular, horizontal cylindrical, vertical cylindrical, or spherical.
7. For rectangular tanks,
   $r_{1i}, r_{2i}, C_{1i}, C_{2i}, h_i, J_{Fri}$
8. For horizontal cylindrical tanks,
   $b_{Hi}, r_{1i}, \beta_i, R_{Hi}, \varrho_i, h_i, J_{Fri}$
9. For vertical cylindrical tanks,
   $b_{Vi}, r_{1i}, u_i, R_{Vi}, h_i$
10. For spherical tanks,
    β_i, R_i, ϱ_i

[illegible]

[illegible] gular tank,

r_{3i} [illegible] : If $r_{1i} = r_{2i}$, $r_{3i} = 1$
If $r_{1i} < r_{2i}$, $r_{3i} = r_{1i}/r_{2i}$
If $r_{1i} > r_{2i}$, $r_{3i} = r_{2i}/r_{1i}$

$$ar_{usi} = (2s-1)\pi r_{ui} \quad (u = 1,2,3;\ s = 1,2,\ldots w_i)$$

$$\sinh ar_{usi} = \frac{1}{2}\left(e^{ar_{usi}} - e^{-ar_{usi}}\right)$$

$$\cosh ar_{usi} = \frac{1}{2}\left(e^{ar_{usi}} + e^{-ar_{usi}}\right)$$

$$\tanh ar_{usi} = \frac{\sinh ar_{usi}}{\cosh ar_{usi}}$$

$$\tanh \frac{ar_{usi}}{2} = \frac{\sinh ar_{usi}}{1 + \cosh ar_{usi}}$$

$$\omega_{ki} = \omega_{usi} = \sqrt{g_i (2s-1)\frac{\pi}{C_{ui}} \tanh ar_{usi}} \qquad (u = 1,2)$$

$$m_{ki} = m_{usi} = M_{Fi} \frac{8 \tanh ar_{usi}}{(2s-1)^3 \pi^3 r_{ui}} \qquad (u = 1,2)$$

$$Z_{ki} = Z_{usi} = \frac{h_i}{2}\left[1 - \frac{4 \tanh \frac{ar_{usi}}{2}}{ar_{usi}}\right] \qquad (u = 1,2)$$

$$J'_{F1i} = J_{F1i}\left\{1 - \frac{4}{1+(r_{2i})^2} + \frac{768}{r_{2i}\left[1+(r_{2i})^2\right]\pi^5}\sum_{s=1}^{w_i} \frac{\tanh \frac{ar_{2si}}{2}}{(2s-1)^5}\right\}$$

$$J'_{F2i} = J_{F2i}\left\{1 - \frac{4}{1+(r_{1i})^2} + \frac{768}{r_{1i}\left[1+(r_{1i})^2\right]\pi^5}\sum_{s=1}^{w_i} \frac{\tanh \frac{ar_{1si}}{2}}{(2s-1)^5}\right\}$$

$$J'_{r3i} = J_{r3i}\left\{1 - \frac{4}{1+(r_{3i})^2} + \frac{768}{r_{3i}\left[1+(r_{3i})^2\right]\pi^6}\sum_{s=1}^{\varpi_i}\frac{\tanh\frac{ar_{3si}}{2}}{(2s-1)^5}\right\}$$

Suffix $k = 4(i-1) + u + 2(s-1)$ [illegible]

$$\Lambda^{rt}_{ki} = m_{ki} Z_{ki} \quad \text{when } r = 3 \text{ and } t = u$$

$$= 0 \quad \text{otherwise } (r, t = 1, 2, 3)$$

$$\mu_{jk} = m_{ki} \quad \text{when } j = k$$

$$= 0 \quad \text{when } j \neq k$$

2. For horizontal cylindrical tanks,

$$ar_{si} = s\pi r_{1i} \quad (s = 1, 2, \ldots \varpi_i)$$

$$\sinh ar_{si} = \tfrac{1}{2}\left(e^{ar_{si}} - e^{-ar_{si}}\right)$$

$$\cosh ar_{si} = \tfrac{1}{2}\left(e^{ar_{si}} + e^{-ar_{si}}\right)$$

$$\tanh ar_{si} = \frac{\sinh ar_{si}}{\cosh ar_{si}}$$

A_{s1}, B_{s1}, and $\sqrt{\lambda_{s1}}$ versus sin β_1

sin β_1	A_{11}	A_{21}	A_{31}	B_{11}	B_{21}	B_{31}	$\sqrt{\lambda_{11}}$	$\sqrt{\lambda_{21}}$	$\sqrt{\lambda_{31}}$
[illegible]	.333	.088	.145	.333	0	0	1.0000	2.4495	3.8730
[illegible]	.343	.139	.175	.338	.002	0	1.0100	2.3800	3.5400
-.80	.355	.192	.207	.344	.005	.001	1.0222	2.3195	3.2939
-.70	.367	.240	.242	.350	.009	.002	1.0400	2.2750	3.1400
-.60	.380	.285	.279	.356	.013	[illegible]	[illegible]	[illegible]	3.0216
-.50	.394	.325	.314	.362	.017	.004	1.0550	2.2000	2.9400
-.40	.408	.362	.352	.368	.022	.005	1.0793	2.1771	2.8862
-.20	.438	.430	.420	.382	.028	.010	1.1176	2.1564	2.8266
.0	.475	.502	.486	.397	.040	.014	1.[illegible]62	2.1679	2.8213
.20	.520	.560	.551	.413	.048	.019	1.2300	2.2158	2.8688
.30	.544	.592	.584	.422	.053	.022	1.2700	2.2500	2.9200
.40	.570	.623	.618	.432	.059	.026	1.3198	2.3108	2.9816
.50	.602	.659	.652	.443	.065	.029	1.3800	2.3800	3.0800
.60	.635	.698	.688	.453	.071	.032	1.4594	2.4940	3.2062
.70	.674	.741	.731	.467	.078	.035	1.5700	2.6400	3.[illegible]
.75	.698	.767	.755	.474	.082	.037	1.6500	2.7500	3.5300
.80	.728	.791	.783	.481	.087	.040	1.7435	2.9017	3.7202
.85	.762	.826	.816	.493	.093	.044	1.8900	3.0800	3.9500
.90	.808	.867	.858	.508	.101	.047	2.1300	3.4300	4.3300
.95	.875	.919	.919	.528	.110	.053	2.4800	4.0000	4.9800
1.00	1.000	1.000	1.000	.558	.121	.058	3.5000	5.5000	7.0000

$$\tanh \frac{ar_{si}}{2} = \frac{\sinh ar_{si}}{1 + \cosh ar_{si}}$$

$$\omega_{ki} = \omega_{1si} = \sqrt{\frac{g_i s\pi}{\ell_{ri}} \tanh ar_{si}}$$

$$m_{ki} = m_{1si} = M_{Fi} \frac{8 \tanh ar_{si}}{\pi^3 s^3 r_{1i}}$$

$$Z_{ki} = Z_{1si} = \frac{h_i}{2} \left(1 - \frac{4 \tanh \frac{ar_{si}}{2}}{ar_{si}} \right)$$

$$J'_{F1i} = 0.$$

$$J'_{F2i} = J_{F2i}\left\{1 - \frac{4}{1+(r_{1i})^2} + \frac{768}{r_{1i}\left[1+(r_{1i})^2\right]\pi^6} \sum_{s=1}^{w_i} \frac{\tanh \frac{ar_{si}}{2}}{s^5}\right\}$$

$$J'_{F3i} = J_{F3i}\left\{1 - \frac{4}{1+(r_{1i})^2} + \frac{768}{r_{1i}\left[1+(r_{1i})^2\right]\pi^6} \sum_{s=1}^{w_i} \frac{\tanh \frac{ar_{si}}{2}}{s^5}\right\}$$

$$m_{ki} = m_{2si} = \frac{2 \rho_i b_{ki} (R_{ki})^2 A_{si} \cos\beta_i}{\lambda_{si}} \qquad (s = 1, 2, \ldots w_i)$$

$$\omega_{ki} = \omega_{2si} = \sqrt{g_i / R_{ki}} \sqrt{\lambda_{si}}$$

$$m'_{ki} = m'_{2si} = 2 \rho_i b_{ki} (R_{ki})^2 B_{si} \cos^2\beta_i$$

$$r_{2i} = 1 + \sin\beta_i$$

If $r_{1i} = 0$, then m_{ki}, , J'_{F1i}, m'_{ki}, and $r_{2i} = 0$.

Suffix $k = 4(i-1) + u + 2(s-1) \qquad (u = 1, 2)$

$$\Lambda_k^{r\sigma} = \eta_{ki} Z_{ki} \quad \text{when } r = 3 \text{ and } \sigma = u = 1$$

$$= 0 \quad \text{otherwise } (r, \sigma = 1, 2, 3)$$

$$\mu_{jk} = m_{ki} \quad \text{when } j = k$$

$$= 0 \quad \text{when } j \neq k.$$

3. For vertical cylindrical tanks,

$$\gamma_1 = 1.84119, \quad \gamma_2 = 5.33144, \quad \gamma_3 = 8.53631$$

$$ar_{si} = \gamma_s r_{ii} \quad (s = 1, 2, 3)$$

$$\sinh ar_{si} = \frac{1}{2}\left(e^{ar_{si}} - e^{-ar_{si}}\right)$$

$$\cosh ar_{si} = \frac{1}{2}\left(e^{ar_{si}} + e^{-ar_{si}}\right)$$

$$\operatorname{csch} ar_{si} = \frac{1}{\sinh ar_{si}}$$

$$\tanh ar_{si} = \frac{\sinh ar_{si}}{\cosh ar_{si}}$$

$$\tanh \frac{ar_{si}}{2} = \frac{\sinh ar_{si}}{1 + \cosh ar_{si}}$$

$$J'_{F1i} = J'_{F2i} = M_{Fi}\left[\frac{(h_i)^2}{12} - \frac{3(R_{vi})^2}{4} + \frac{16(R_{vi})^2}{r_{ii}}\sum_{s=1}^{3}\frac{\tanh\frac{a\eta_{si}}{2}}{\gamma_s^3(\gamma_s^2\ \cdot)}\right]$$

If $\quad = 0$, J_{F1i} $\quad J_{F2i} = 0$.

$J'_{F3i} = 0$

$$A'_{si} = \frac{s\pi}{r_{ii}}$$

$\Psi_{si} = \psi(A'_{si})$ according to the following table.

If $r_{ii} = 0$, $\Psi_{si} = 0$

Ψ_{si} versus A'_{si}

A'_{si}	Ψ_{si}
0	1.0
.25	.982
.41	.956
1.00	.791
1.46	.661
1.88	.569
2.18	.506
2.49	.444
3.00	.372
3.77	.290
4.00	.268
4.45	.233
5.00	.203
5.79	.180
6.82	.152
8.60	.122
10.00	.101

If $A'_{si} > 10$, $\Psi_{si} = 0$

When $u_1 > 1/\sqrt{2}$

$$F'_{k0i} = \frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} f_{ki} \, dx'_i \qquad (k > 40)$$

$$F'_{ksi} = 2 \frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} f_{ki} \cos \left[s\pi \frac{b_{vi}}{h_i} \left(\tfrac{1}{2} + x'_i \right) \right] dx'_i$$

$f'_{k0i} = f'_{ki}$ evaluated for $x'_i = -\frac{1}{2}$,

$f'_{kri} = f'_{ki}$ evaluated for $x'_i = \frac{h_i}{b_{vi}} - \frac{1}{2}$,

$$G_{ki} = \frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} x'_i f_{ki} \, dx'_i$$

$$G'_{ki} = b_{vi} \int_{-\frac{1}{2}}^{0} (f'_{ki})^2 \, dx'_i$$

$$G''_{ki} = b_{vi} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} (f'_{ki})^2 \, dx'_i$$

$$G'''_{ki} = b_{vi} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} x'_i (f'_{ki})^2 \, dx'_i$$

When $u_i < -1/\sqrt{2}$

$$F'_{kci} = \frac{b_{vi}}{h_i}\int_{\frac{1}{2}-\frac{h_i}{b_{vi}}}^{\frac{1}{2}} f_{ki}\, dX'_i$$

$$F'_{ksi} = 2\,\frac{b_{vi}}{h_i}\int_{\frac{1}{2}-\frac{h_i}{b_{vi}}}^{\frac{1}{2}} f_{ki} \cos\left[s\pi \frac{b_{vi}}{h} \left(\tfrac{1}{2} - X'_i\right)\right] dX'_i$$

$f'_{kbi} = f'_{ki}$ evaluated for $X'_i = \frac{1}{2}$,

$f'_{kti} = f'_{ki}$ evaluated for $X'_i = \frac{1}{2} - \frac{h_i}{b_{vi}}$,

$$G_{ki} = -\frac{b_{vi}}{h_i}\int_{\frac{1}{2}-\frac{h_i}{b_{vi}}}^{\frac{1}{2}} X'_i f_{ki}\, dX'_i$$

$$G'_{ki} = b_{vi}\int_{0}^{\frac{1}{2}} (f'_{ki})^2\, dX'_i$$

$$G''_{ki} = b_{vi}\int_{\frac{1}{2}-\frac{h_i}{b_{vi}}}^{\frac{1}{2}} (f'_{ki})^2\, dX'_i$$

$$G'''_{ki} = -b_{vi}\int_{\frac{1}{2}-\frac{h_i}{b_{vi}}}^{\frac{1}{2}} X'_i (f'_{ki})^2\, dX'_i$$

$$Y_{ks_1} = \sum_{p=0}^{3} \left\{ (-1)^p F_{kp_1} \Big/ \left[(\gamma_s)^2 + \left(\frac{p\pi}{r_{11}}\right)^2 \right] \right\}$$

$$Y'_{ks_1} = \sum_{p=0}^{3} \left\{ F'_{kp_1} \Big/ \left[(\gamma_s)^2 + \left(\frac{p\pi}{r_{11}}\right)^2 \right] \right\}$$

$$u''_{1k1} = M_{F_1} \left[F'_{k01} - \frac{R_{v1}}{4 r_{11}} \left(f'_{k01} - f'_{k01} \right) \right]$$

$$u''_{1,s+3,1} = \frac{M_{Fi}}{\gamma_s^2 r_{1i}}$$

$$u''_{2ki} = M_{F_1} \left[\frac{b_{v1} - h_1}{2} F'_{k01} + b_{v1} G_{k1} + \frac{4h_i}{\pi^2} \sum_{s=1}^{2} \frac{\psi_{2s-1,1} F'_{k,2s-1,1}}{(2s-1)^2} \right.$$

$$\left. + \left(f'_{kti} + f'_{kb1} \right) \left(\frac{(h_1)^2}{12} - \frac{x_{vi}^2}{8} + \frac{8h_1^2}{\pi^4} \sum_{s=1}^{2} \frac{\psi_{2s-1,1}}{(2s-1)^4} \right) \right]$$

$$u''_{2,s+3,1} = \frac{M_{F_1} R_{vi}}{\gamma_s^2} \left[\frac{2 \tanh \frac{a r_{si}}{2}}{a r_{si}} - \frac{1}{2} \right]$$

$$u'_{kki} = M_{F_1} \left\{ (F'_{k01})^2 + \frac{1}{2} \sum_{s=1}^{3} \psi_{si} (F'_{ksi})^2 - \frac{2 R_{v1}}{r_{11}} \sum_{s=1}^{3} \frac{f'_{kti} Y'_{ksi} - f'_{kbi} Y^*_{ksi}}{\gamma_s^2 - 1} \right.$$

$$+ \frac{E'_{oi} R_{vi}}{4 r_{1i}} \left(f'_{kbi} - f'_{kti} \right) + \frac{h_i}{\pi^2} \sum_{s=1}^{3} F'_{ksi} \frac{1 - \psi_{si}}{s^2} \left[(-1)^{s+1} f'_{ksi} + f'_{kbi} \right]$$

$$\left. + \frac{2 R_{vi}^2}{r_{11}} \sum_{s=1}^{3} \frac{\operatorname{csch} a r_{si}}{\gamma_s^2 (\gamma_s^2 - 1)} \left[\left((f'_{kti})^2 + (f'_{kbi})^2 \right) \cosh a r_{si} - 2 f'_{kt} f'_{kbi} \right] \right\}$$

$$u_{k,s+3,i} = \frac{M_{Fi}}{r_{ii}} \left\{ Y'_{ksi} - \frac{R_{vi}}{\gamma_s^3} (\operatorname{csch} ar_{si}) \left[f'_{kvi} \cosh (ar_s - f'_{\dots}) \right] \right\}$$

$$u'_{s+3,s+3,i} = \frac{M_{Fi}}{2 r_{ii} \gamma_s^3} (\gamma_s^2 - 1) / \tanh ar_{si}$$

$$\omega^2_{s+3,i} = \frac{\gamma_i \gamma_s}{R_{vi}} \tanh ar_{si}$$

If $r_{ii} = 0$, $M_{Fi} = 0$, the u'_{mni} equal zero.

$$\Lambda'^{rt}_{ki} = 0.$$

$u_{jk} = u'_{s+3,s+3,i}$ when $j = k = 4(i-1) + u + 2(s-1)$

$= 0$ when $j \neq k$

$u'_{jk} = 0$ when both j and $k > 40$ but $j \neq k$.

4. For spherical tanks,

C_{si}, D_{si}, and $\sqrt{\lambda'_{si}}$ versus $\sin \beta_i$

$\sin \beta_i$	C_{1i}	C_{2i}	C_{3i}	D_{1i}	D_{2i}	D_{3i}	$\sqrt{\lambda'_{1i}}$	$\sqrt{\lambda'_{2i}}$	$\sqrt{\lambda'_{3i}}$
-1.00	.245	.122	.080	.251	0	0	1.0000	2.6500	4.1200
- .90	.250	.164	.134	.253	.002	.001	1.0149	2.5787	3.7855
- .80	.255	.204	.184	.256	.003	[illegible]	1.0384	2.5080	3.4900
- .70	.260	.242	.220	.259	.006	.003	1.0650	2.4600	3.2700
- .60	.268	.276	.272	.262	.011	.004	[illegible]	2.3937	3.1575
- .50	.273	.307	.302	.266	.012	.005	1.0957	2.3565	3.0750
- .40	.281	.350	.350	.270	.016	.006	1.1225	2.3238	2.9983
- .20	.305	.415	.418	.278	.023	.011	1.1790	2.2913	2.9274
.0	.335	.475	.485	.290	.032	.015	1.2490	2.2956	2.9138
.20	.370	.535	.545	.301	.039	.018	1.3379	2.3452	2.9648
.30	.392	.565	.578	.309	.044	.020	1.4000	2.3973	[illegible]
.40	.420	.595	.612	.318	.050	.022	1.4629	2.4495	3.0871
.50	.453	.632	.650	.328	.058	.024	1.5800	2.5600	3.2200
.60	.492	.672	.692	.338	.066	.027	1.6643	2.6571	3.3377
.70	.543	.718	.740	.353	.074	.030	1.8300	2.8200	3.5900
.75	.578	.742	.768	.363	.081	.032	1.9200	2.9300	3.7300
.80	.620	.770	.794	.373	.087	.033	2.0784	3.1321	3.9281
.85	.671	.804	.826	.388	.094	.037	2.1800	3.3700	4.1900
.90	.732	.847	.868	.407	.104	.040	2.4200	3.7700	4.6400
.95	.818	.907	.922	.433	.117	.043	2.7700	4.4300	5.3000
1.00	1.000	1.000	1.000	.470	.134	.047	4.0000	6.0000	7.5000

$$m_{si} = m_{usi} = \pi \rho_i R_i^3 \frac{C_{si}}{\lambda'_{si}} \cos^2 \beta_i \quad (s = 1, 2, \ldots)$$

$$\omega_{si} = \omega_{usi} = \sqrt{g_i / R_i} \; \sqrt{\lambda'_{si}}$$

$$m'_{si} = m'_{usi} = \pi \rho_i R_i^3 D_{si} \cos^3 \beta_i$$

$$J'_{F1i} = J'_{F2i} = J'_{F3i} = 0$$

If $\beta_i = \pm\frac{\pi}{2}$, then

m_{ki}, $\dot{k}_{ki}$, and m'_{ki} equal zero.

$$\Lambda''_{ki} = 0$$

$$\mu_{jk} = m_{ki} \quad \text{when } j = k + 4(i-1) + u + 2(s-1)$$

$$= 0 \quad \text{when } j \neq k.$$

5. For all tanks,

$$\begin{bmatrix} I_{F1i} & -P_{F3i} & -P_{F2i} \\ -P_{F3i} & I_{F2i} & -P_{F1i} \\ -P_{F2i} & -P_{F1i} & I_{F3i} \end{bmatrix} =$$

$$\begin{bmatrix} \ell^1_{1i} & \ell^1_{2i} & \ell^1_{3i} \\ \ell^2_{1i} & \ell^2_{2i} & \ell^2_{3i} \\ \ell^3_{1i} & \ell^3_{2i} & \ell^3_{3i} \end{bmatrix} \begin{bmatrix} J'_{F1i} & 0 & 0 \\ 0 & J'_{F2i} & 0 \\ 0 & 0 & J'_{F3i} \end{bmatrix} \begin{bmatrix} \ell^1_{1i} & \ell^2_{1i} & \ell^3_{1i} \\ \ell^1_{2i} & \ell^2_{2i} & \ell^3_{2i} \\ \ell^1_{3i} & \ell^2_{3i} & \ell^3_{3i} \end{bmatrix}$$

$$F_{11} = \sum_{i=1}^{T} \left[M_{Fi} \left(\mathfrak{z}^2_{Fi} \mathfrak{z}^2_{Fi} + \mathfrak{z}^3_{Fi} \mathfrak{z}^3_{Fi} \right) + I_{F1i} \right]$$

$$F_{12} = - \sum_{i=1}^{T} \left(M_{Fi} \mathfrak{z}^1_{Fi} \mathfrak{z}^2_{Fi} + P_{F3i} \right)$$

$$F_{13} = - \sum_{i=1}^{T} \left(M_{Fi} \mathfrak{z}^1_{Fi} \mathfrak{z}^3_{Fi} + P_{F2i} \right)$$

$$F_{21} = F_{12}$$

$$F_{22} = \sum_{i=1}^{T} \left[M_{Fi} \left(\mathfrak{z}^3_{Fi} \mathfrak{z}^3_{Fi} + \mathfrak{z}^1_{Fi} \mathfrak{z}^1_{Fi} \right) + I_{F2i} \right]$$

$$F_{23} = - \sum_{i=1}^{T} \left(M_{Fi} \mathfrak{z}^2_{Fi} \mathfrak{z}^3_{Fi} + P_{F1i} \right)$$

$$F_{31} = F_{13} \ , \quad F_{32} = F_{23}$$

$$F_{33} = \sum_{i=1}^{T} \left[M_{Fi} \left(\mathfrak{z}^1_{Fi} \mathfrak{z}^1_{Fi} + \mathfrak{z}^2_{Fi} \mathfrak{z}^2_{Fi} \right) + I_{F3i} \right]$$

$$\Gamma'_{F1i} = \left(J'_{F2i} + J'_{F3i} - J'_{F1i} \right) / 2$$

$$\Gamma'_{F2i} = \left(J'_{F3i} + J'_{F1i} - J'_{F2i} \right) / 2$$

$$\Gamma'_{F3i} = \left(J'_{F1i} + J'_{F2i} - J'_{F3i} \right) / 2$$

$$\underset{\sim}{H}F^{t}_{k} = \sum_{i=1}^{T} M_{Fi} \mathfrak{z}^{t}_{ki}$$

$$LF_i^{st} = \sum_{i=1}^{T} M_{Fi}\, \zeta^{s}_{Fi}\, \zeta^{t}_{ki}$$

$$\beta_{ki}^{r} = \sum_{r=1} \ell_{ri}\, B'_{ki}$$

$$\zeta^{r}_{ki} = \chi^{r}_{ki} + \left(\ell^{r}_{1i} B'^{2}_{k} - \ell^{r}_{2i} B'^{1}_{ki}\right) \bar{Z}$$

$$\{\Lambda F_k\} = \sum_{i=1}^{T} \begin{bmatrix} \ell^{1}_{1i} & \ell^{1}_{2i} & \ell^{1}_{3i} \\ \ell^{2}_{1i} & \ell^{2}_{2i} & \ell^{2}_{3i} \\ \ell^{3}_{1i} & \ell^{3}_{2i} & \ell^{3}_{3i} \end{bmatrix} \begin{Bmatrix} -\Lambda'^{32}_{ki} \\ \Lambda'^{31}_{ki} \\ 0 \end{Bmatrix}$$

$$\{NF_k\} = \sum_{i=1}^{T} \begin{bmatrix} \ell^{1}_{1i} & \ell^{1}_{2i} & \ell^{1}_{3i} \\ \ell^{2}_{1i} & \ell^{2}_{2i} & \ell^{2}_{3i} \\ \ell^{3}_{1i} & \ell^{3}_{3i} & \ell^{3}_{3i} \end{bmatrix} \begin{bmatrix} J'_{F1i} & 0 & 0 \\ 0 & J'_{F2i} & 0 \\ 0 & 0 & J'_{F3i} \end{bmatrix} \begin{Bmatrix} B'^{1}_{ki} \\ B'^{2}_{ki} \\ B'^{3}_{ki} \end{Bmatrix}$$

$$\Sigma F_j = \sum_{i=1}^{T} \begin{bmatrix} \ell^{1}_{1i} & \ell^{1}_{2i} & \ell^{1}_{3i} \\ \ell^{2}_{1i} & \ell^{2}_{2i} & \ell^{2}_{3i} \\ \ell^{3}_{1i} & \ell^{3}_{2i} & \ell^{3}_{3i} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Lambda'^{31}_{ji} & \Lambda'^{32}_{ji} & 0 \end{bmatrix} \begin{bmatrix} \ell^{1}_{1i} & \ell^{2}_{1i} & \ell^{3}_{1i} \\ \ell^{1}_{2i} & \ell^{2}_{2i} & \ell^{3}_{2i} \\ \ell^{1}_{3i} & \ell^{2}_{3i} & \ell^{3}_{3i} \end{bmatrix}$$

$$\Theta F_j = \sum_{i=1}^{T} \begin{bmatrix} \ell^{1}_{1i} & \ell^{1}_{2i} & \ell^{1}_{3i} \\ \ell^{2}_{1i} & \ell^{2}_{2i} & \ell^{2}_{3i} \\ \ell^{3}_{1i} & \ell^{3}_{2i} & \ell^{3}_{3i} \end{bmatrix} \begin{bmatrix} 0 & \beta'^{3}_{ji}\Gamma'_{F1i} & -\beta'^{2}_{ji}\Gamma'_{F1i} \\ -\beta'^{3}_{ji}\Gamma'_{F2i} & 0 & \beta'^{1}_{ji}\Gamma'_{F2i} \\ \beta'^{2}_{ji}\Gamma'_{F3i} & -\beta'^{1}_{ji}\Gamma'_{F3i} & 0 \end{bmatrix} \begin{bmatrix} \ell^{1}_{1i} & \ell^{2}_{1i} & \ell^{3}_{1i} \\ \ell^{1}_{2i} & \ell^{2}_{2i} & \ell^{3}_{2i} \\ \ell^{1}_{3i} & \ell^{2}_{3i} & \ell^{3}_{3i} \end{bmatrix}$$

$$\underset{\sim}{A}F_{jk} = -\beta_{ji}'^{11}\Lambda_{ki}'^{32} + \beta_{ji}'^{12}\Lambda_{ki}'^{31}$$

$$-\beta_{ki}'^{11}\Lambda_{ji}'^{32} + \beta_{ki}'^{12}\Lambda_{ji}'^{31} \qquad (j, k \le 40)$$

$$\underset{\sim}{\eta}F_{jk} = \sum_{i=1}^{T} M_{Fi} \sum_{r=1}^{3} \gamma_{ji}^{r} \gamma_{ki}^{r} \qquad (j, k > 40)$$

$$\underset{\sim}{H}F_{jk} = \sum_{i=1}^{T} \left[\beta_{ji}'^{1} \; \beta_{ji}'^{2} \; \beta_{ji}'^{3}\right] \begin{bmatrix} J'_{F1i} & 0 & 0 \\ 0 & J'_{F2i} & 0 \\ 0 & 0 & J'_{F3i} \end{bmatrix} \begin{Bmatrix} \beta_{ki}'^{1} \\ \beta_{ki}'^{2} \\ \beta_{ki}'^{3} \end{Bmatrix}$$

6. For structure, including tanks but not fluid,

If $P_i > 1$, $\Lambda_{ki}'^{rs} = \sum_{h=1}^{P_i} m_{ih} \, \nu_{ih}'^{r} \, \sigma_{kih}'^{s}$

$(r, s = 1, 2, 3, \; k = 1, 2, \ldots n; \; i = 1, 2, \ldots S)$

If $P_i = 1$, $\Lambda_{ki}'^{rs} = 0$.

$$\Gamma_{11i}' = (J_{2i} + J_{3i} - J_{1i})/2$$

$$\Gamma_{22i}' = (J_{3i} + J_{1i} - J_{2i})/2$$

$$\Gamma'_{33i} = (J_{1i} + J_{2i} - J_{3i})/2$$

If P . . or if σ'^{r} . . .

$$\beta'^{s}_{Ki} = \sum_{r=1}^{3} e^{r}_{si} \beta^{r}_{Ki}$$

$$\gamma^{r} = \lambda^{r} - \begin{vmatrix} e^{r}_{1i} & \beta'^{1}_{Ki} & \rho'^{1}_{i} \\ e^{r}_{2i} & \beta'^{2}_{Ki} & \rho'^{2}_{i} \\ e^{r}_{3i} & \beta'^{3}_{Ki} & \rho'^{3}_{i} \end{vmatrix}$$

If $P_1 > 1$ and if $\sigma'^{r}_{Kih} \neq 0$,

$$\gamma^{r}_{Ki} = \sum_{s=1}^{3} e^{r}_{si} \gamma'^{s}_{Ki} .$$

$$\underset{\smile}{H}\underset{\smile}{S}^{st}_{K} = \sum_{i=1}^{5} m_i \gamma^{t}_{Ki}$$

$$\underset{\smile}{L}\underset{\smile}{S}^{st}_{K} = \sum_{i=1}^{5} m_i \gamma^{s}_{i} \gamma^{t}_{Ki}$$

$$\left\{ \underset{\smile}{\Lambda S}_{K} \right\} = \sum_{i=1}^{5} \begin{bmatrix} e^{1}_{1i} & e^{1}_{2i} & e^{1}_{3i} \\ e^{2}_{1i} & e^{2}_{2i} & e^{2}_{3i} \\ e^{3}_{1i} & e^{3}_{2i} & e^{3}_{3i} \end{bmatrix} \begin{Bmatrix} \Lambda'^{23}_{Ki} - \Lambda'^{32}_{Ki} \\ \Lambda'^{31}_{Ki} - \Lambda'^{13}_{Ki} \\ \Lambda'^{12}_{Ki} - \Lambda'^{21}_{Ki} \end{Bmatrix}$$

$$
\{NS_k\} = \sum_{i=1}^{5} \begin{bmatrix} e^1_{1i} & e^1_{2i} & e^1_{3i} \\ e^2_{1i} & e^2_{2i} & e^2_{3i} \\ e^3_{1i} & e^3_{2i} & e^3_{3i} \end{bmatrix} \begin{bmatrix} J_{1i} & -K_{3i} & K_{2i} \\ -K_{3i} & J_{2i} & -K_{1i} \\ -K_{2i} & -K_{1i} & J_{3i} \end{bmatrix} \begin{Bmatrix} \beta'^{1}_{ki} \\ \beta'^{2}_{ki} \\ \beta'^{3}_{ki} \end{Bmatrix}
$$

$$
\Sigma S_j = \sum_{i=1}^{5} \begin{bmatrix} e^1_{1i} & e^1_{2i} & e^1_{3i} \\ e^2_{1i} & e^2_{2i} & e^2_{3i} \\ e^3_{1i} & e^3_{2i} & e^3_{3i} \end{bmatrix} \begin{bmatrix} (\Lambda'^{22}_{ji} + \Lambda'^{33}_{ji}) & -\Lambda'^{12}_{ji} & -\Lambda'^{13}_{ji} \\ -\Lambda'^{21}_{ji} & (\Lambda'^{33}_{ji} + \Lambda'^{11}_{ji}) & -\Lambda'^{23}_{ji} \\ -\Lambda'^{31}_{ji} & -\Lambda'^{32}_{ji} & (\Lambda'^{11}_{ji} + \Lambda'^{22}_{ji}) \end{bmatrix} \begin{bmatrix} e^1_{1i} & e^2_{1i} & e^3_{1i} \\ e^1_{2i} & e^2_{2i} & e^3_{2i} \\ e^1_{3i} & e^2_{3i} & e^3_{3i} \end{bmatrix}
$$

$$
\Theta S_j = \sum_{i=1}^{5} \begin{bmatrix} e^1_{1i} & e^1_{2i} & e^1_{3i} \\ e^2_{1i} & e^2_{2i} & e^2_{3i} \\ e^3_{1i} & e^3_{2i} & e^3_{3i} \end{bmatrix}
$$

$$
\begin{bmatrix} (\beta'^{2}_{ji} K_{2i} - \beta'^{3}_{ji} K_{3i}) & (\beta'^{3}_{ji} \Gamma'_{11i} - \beta'^{1}_{ji} K_{2i}) & (\beta'^{1}_{ji} K_{3i} - \beta'^{2}_{ji} \Gamma'_{11i}) \\ (\beta'^{2}_{ji} K_{1i} - \beta'^{3}_{ji} \Gamma'_{22i}) & (\beta'^{3}_{ji} K_{3i} - \beta'^{1}_{ji} K_{1i}) & (\beta'^{1}_{ji} \Gamma'_{22i} - \beta'^{2}_{ji} K_{3i}) \\ (\beta'^{2}_{ji} \Gamma'_{33i} - \beta'^{3}_{ji} K_{1i}) & (\beta'^{3}_{ji} K_{2i} - \beta'^{1}_{ji} \Gamma'_{33i}) & (\beta'^{1}_{ji} K_{1i} - \beta'^{2}_{ji} K_{2i}) \end{bmatrix}
$$

$$
\begin{bmatrix} e^1_{1i} & e^2_{1i} & e^3_{1i} \\ e^1_{2i} & e^2_{2i} & e^3_{2i} \\ e^1_{3i} & e^2_{3i} & e^3_{3i} \end{bmatrix}
$$

$$
\Phi_{jk} = 0.
$$

$$\underset{\sim}{\Delta S}_{jk} = \sum_{i=1}^{s} \Big[\beta'^{1}_{ji} \left(\Lambda'^{23}_{ki} - \Lambda'^{32}_{ki} \right) + \beta'^{2}_{ji} \left(\Lambda'^{31}_{ki} - \Lambda'^{13}_{ki} \right) + \beta'^{3}_{ji} \left(\Lambda'^{12}_{ki} - \Lambda'^{21}_{ki} \right)$$

$$+ \beta'^{1}_{ki} \left(\Lambda'^{23}_{ji} - \Lambda'^{32}_{ji} \right) + \beta'^{2}_{ki} \left(\Lambda'^{31}_{ji} - \Lambda'^{13}_{ji} \right) + \beta'^{3}_{ki} \left(\Lambda'^{12}_{ji} - \Lambda'^{21}_{ji} \right) \Big]$$

$$\underset{\sim}{\eta S}_{jk} = \sum_{i=1}^{s} m_i \sum_{r=1}^{3} f^{r}_{ji} \, f^{r}_{ki}$$

$$\underset{\sim}{HS}_{jk} = \sum_{i=1}^{s} \left[\beta'^{1}_{ji} \; \beta'^{2}_{ji} \; \beta'^{3}_{ji} \right] \begin{bmatrix} J_{1i} & -K_{3i} & -K_{2i} \\ -K_{3i} & J_{2i} & -K_{1i} \\ -K_{2i} & -K_{1i} & J_{3i} \end{bmatrix} \begin{Bmatrix} \beta'^{1}_{ki} \\ \beta'^{2}_{ki} \\ \beta'^{3}_{ki} \end{Bmatrix}$$

$$u_{jk} = \sum_{i=1}^{s} \sum_{h=1}^{p_i} m_{ih} \sum_{r=1}^{3} \sigma'^{r}_{jih} \, \sigma'^{r}_{kih}$$

$$S'^{rs}_{jki} = \sum_{h=1}^{p_i} m_{ih} \, \sigma'^{r}_{jih} \, \sigma'^{s}_{kih}$$

7. For structure and tanks, including fluid,

$$I_{rs} = S_{rs} + F_{rs} \quad (r, s = 1, 2, 3)$$

$$G_{rs} = \delta_{rs} \sum_{t=1}^{3} I_{tt}/2 - I_{rs},$$

$$\delta_{rs} = 1 \text{ when } r = s$$

$$= 0 \text{ when } r \neq s$$

$$D = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} - m \begin{bmatrix} (\mathfrak{z}_c^2 \mathfrak{z}_c^2 + \mathfrak{z}_c^3 \mathfrak{z}_c^3) & -\mathfrak{z}_c^1 \mathfrak{z}_c^2 & -\mathfrak{z}_c^1 \mathfrak{z}_c^3 \\ -\mathfrak{z}_c^2 \mathfrak{z}_c^1 & (\mathfrak{z}_c^3 \mathfrak{z}_c^3 + \mathfrak{z}_c^1 \mathfrak{z}_c^1) & -\mathfrak{z}_c^2 \mathfrak{z}_c^3 \\ -\mathfrak{z}_c^3 \mathfrak{z}_c^1 & -\mathfrak{z}_c^3 \mathfrak{z}_c^2 & (\mathfrak{z}_c^1 \mathfrak{z}_c^1 + \mathfrak{z}_c^2 \mathfrak{z}_c^2) \end{bmatrix}$$

$$\{b_k\} = \begin{Bmatrix} b_k^1 \\ b_k^2 \\ b_k^3 \end{Bmatrix} \qquad (k = 1, 2, \ldots n)$$

$$\{\Lambda_k\} = \{\Lambda F_k\} + \{\Lambda S_k\}$$

$$\{N_k\} = \{NF_k\} + \{NS_k\}$$

$$H_k^t = \sum_{i=1}^{s} m_i \mathfrak{z}_{ki}^t + \sum_{i=1}^{T} M_{Fi} \mathfrak{z}_{ki}^t = HF_k^t + HS_k^t$$

$$L_k^{st} = \sum_{i=1}^{s} m_i \mathfrak{z}_i^s \mathfrak{z}_{ki}^t + \sum_{i=1}^{T} M_{Fi} \mathfrak{z}_{Fi}^s \mathfrak{z}_{ki}^t = LF_k^{st} + LS_k^{st}$$

$$\{\epsilon_k\} = \{\Lambda_k\} + \{N_k\} + \begin{Bmatrix} L_k^{23} - L_k^{32} \\ L_k^{31} - L_k^{13} \\ L_k^{12} - L_k^{21} \end{Bmatrix}$$

$$\{E_k\} = \begin{Bmatrix} \mathfrak{z}_c^2 H_k^3 - \mathfrak{z}_c^3 H_k^2 \\ \mathfrak{z}_c^3 H_k^1 - \mathfrak{z}_c^1 H_k^3 \\ \mathfrak{z}_c^1 H_k^2 - \mathfrak{z}_c^2 H_k^1 \end{Bmatrix} - \{\epsilon_k\}$$

$$\{b_k\} = D^{-1} \{E_k\}$$

$$\begin{Bmatrix} C_k^1 \\ C_k^2 \\ C_k^3 \end{Bmatrix} = \begin{Bmatrix} z_c^2 b_k^3 - z_c^3 b_k^2 \\ z_c^3 b_k^1 - z_c^1 b_k^3 \\ z_c^1 b_k^2 - z_c^2 b_k^1 \end{Bmatrix} - \frac{1}{m} \begin{Bmatrix} H_k^1 \\ H_k^2 \\ H_k^3 \end{Bmatrix}$$

$$\begin{bmatrix} P_{11j} & P_{12j} & P_{13j} \\ P_{21j} & P_{22j} & P_{23j} \\ P_{31j} & P_{32j} & P_{33j} \end{bmatrix} = m \begin{bmatrix} (z_c^2 C_j^2 + z_c^3 C_j^3) & z_c^1 C_j^2 & -z_c^1 C_j^3 \\ -z_c^2 C_j^1 & (z_c^3 C_j^3 + z_c^1 C_j^1) & -z_c^2 C_j^3 \\ -z_c^3 C_j^1 & -z_c^3 C_j^2 & (z_c^1 C_j^1 + z_c^2 C_j^2) \end{bmatrix}$$

$$- \begin{bmatrix} (b_j^2 G_{13} - b_j^3 G_{12}) & (b_j^3 G_{11} - b_j^1 G_{13}) & (b_j^1 G_{12} - b_j^2 G_{11}) \\ (b_j^2 G_{23} - b_j^3 G_{22}) & (b_j^3 G_{21} - b_j^1 G_{23}) & (b_j^1 G_{22} - b_j^2 G_{21}) \\ (b_j^2 G_{33} - b_j^3 G_{32}) & (b_j^3 G_{31} - b_j^1 G_{33}) & (b_j^1 G_{32} - b_j^2 G_{31}) \end{bmatrix}$$

$$+ \begin{bmatrix} (L_j^{22} + L_j^{33}) & -L_j^{12} & -L_j^{13} \\ -L_j^{21} & (L_j^{33} + L_j^{11}) & -L_j^{23} \\ -L_j^{31} & -L_j^{32} & (L_j^{11} + L_j^{22}) \end{bmatrix} + \Sigma \underset{\sim}{F}_j + \Sigma \underset{\sim}{S}_j - \Theta \underset{\sim}{F}_j - \Theta \underset{\sim}{S}_j$$

$$(j = 1, 2, \quad n)$$

$$\Delta_{jk} = \Delta \underset{\sim}{F}_{jk} + \Delta \underset{\sim}{S}_{jk}$$

$$\eta_{jk} = \eta \underset{\sim}{F}_{jk} + \eta \underset{\sim}{S}_{jk}$$

$$H_{jk} = H \underset{\sim}{F}_{jk} + H \underset{\sim}{S}_{jk}$$

For fuel $\alpha^r_{k1} = \beta'^r_{k1} + \sum_{s=1}^{3} \ell^s_{r1} b^s_k$

$h^1_{k1} = c^1_k + g^1_{k1} + b^2_k z^3_{r1} - b^3_k z^2_{r1}$

$h^2_{k1} = c^2_k + g^2_{k1} + b^3_k z^1_{r1} - b^1_k z^3_{r1}$

$h^3_{k1} = c^3_k + g^3_{k1} + b^1_k z^2_{r1} - b^2_k z^1_{r1}$

For a spherical tank or for lateral sloshing ($u=2$)

in a horizontal cylindrical tank,

$$\Phi_{jk} = \sum_{r=1}^{3} \ell^r_{21} \left(h^r_{j1}\, m'_{k1} + h^r_{k1}\, m'_{j1} \right)$$

if j or k denotes a fuel slosh mode (j or $k \leq 40$).

$\Phi_{jk} = 0$ if neither j nor k denotes a fuel

slosh mode (j and $k > 40$).

For vertical cylindrical tanks, assuming $u=2$,

$$\Phi_{jk} = \sum_{i=1}^{T}\Big[u'_{jki} + \sum_{r=1}^{3} l^r_{2i}\,(h^r_{ji}\,u'_{1ki} + h^r_k\,u''_{j1i})$$

$$+ \alpha'^{1}_{ji}\,u''_{2ki} + \alpha'^{1}_{k}\,u''_{j2i}\Big]$$

For rectangular tanks and for longitudinal sloshing in horizontal cylindrical tanks, $\Phi_{jk}=0$.

$$d^1_{jk} = c^2_j b^3_k - c^3_j b^2_k + c^2_k b^3_j - c^3_k b^2_j$$

$$d^2_{jk} = c^3_j b^1_k - c^1_j b^3_k + c^3_k b^1_j - c^1_k b^3_j$$

$$d^3_{jk} = c^1_j b^2_k - c^2_j b^1_k + c^1_k b^2_j - c^2_k b^1_j.$$

For structure and tanks, including fuel,

$$M_{jk} = m\sum_{r=1}^{3}\left(c^r_j c^r_k + \bar{z}^r_c d^r_{jk}\right) + \sum_{r=1}^{3}\left(c^r_j H^r_k + c^r_k H^r_j + b^r_j \epsilon^r_k + b^r_k \epsilon^r_j\right)$$

$$+ \eta_{jk} + \Delta_{jk} + H_{jk} + u_{jk} + \Phi_{jk}$$

$$+ \begin{bmatrix} b^1_j & b^2_j & b^3_j \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{Bmatrix} b^1_k \\ b^2_k \\ b^3_k \end{Bmatrix}$$

For fuel,

$$a_{ji}^{\prime 12} = a_{ji}^{\prime 21} = \frac{1}{2}\,\alpha_{ji}^{\prime 3}\left(J_{F2i}^{\prime} - J_{F1i}^{\prime}\right)$$

$$a_{ji}^{\prime 13} = a_{ji}^{\prime 31} = \frac{1}{2}\left[\alpha_{ji}^{\prime 2}\left(J_{F1i}^{\prime} - J_{F3i}^{\prime}\right) + \Lambda_{ji}^{\prime 21}\right]$$

$$a_{ji}^{\prime 23} = a_{ji}^{\prime 32} = \frac{1}{2}\left[\alpha_{ji}^{\prime 1}\left(J_{F3i}^{\prime} - J_{F2i}^{\prime}\right) + \Lambda_{ji}^{\prime 32}\right]$$

$$\boxed{KL,j}_F = \sum_{i=1}^{T}\left[\alpha_{Ki}^{\prime 1}\ \alpha_{Ki}^{\prime 2}\ \alpha_{Ki}^{\prime 3}\right]\begin{bmatrix} 0 & a_{ji}^{\prime 12} & a_{ji}^{\prime 13} \\ a_{ji}^{\prime 21} & 0 & a_{ji}^{\prime 23} \\ a_{ji}^{\prime 31} & a_{ji}^{\prime 32} & 0 \end{bmatrix}\begin{Bmatrix} \alpha_{Li}^{\prime 1} \\ \alpha_{Li}^{\prime 2} \\ \alpha_{Li}^{\prime 3} \end{Bmatrix}$$

$$-\sum_{i=1}^{T}\Big[\left(\alpha_{Ki}^{\prime 3}\Lambda_{Li}^{\prime 31} + \alpha_{Li}^{\prime 3}\Lambda_{Ki}^{\prime 31}\right)\alpha_{ji}^{\prime 1}$$

$$+\left(\alpha_{Ki}^{\prime 3}\Lambda_{Li}^{\prime 32} + \alpha_{Li}^{\prime 3}\Lambda_{Ki}^{\prime 32}\right)\alpha_{ji}^{\prime 2}\Big]$$

For structure, $\alpha_{ji}^{\prime r} = \beta_{ji}^{\prime r} + \sum_{s=1}^{3} e_{ri}^{s}\, b_{j}^{s}$

$$a_{ji}^{\prime 11} = \alpha_{ji}^{\prime 2} K_{2i} - \alpha_{ji}^{\prime 3} K_{3i} - \Lambda_{ji}^{\prime 22} - \Lambda_{ji}^{\prime 33}$$

$$a_{ji}^{\prime 12} = \frac{1}{2}\left[\alpha_{ji}^{\prime 3}\left(J_{2i} - J_{1i}\right) + \alpha_{ji}^{\prime 2} K_{1i} - \alpha_{ji}^{\prime 1} K_{2i} - \Lambda_{ji}^{\prime 12} + \Lambda_{ji}^{\prime 21}\right]$$

$$a'^{13}_{ji} = \tfrac{1}{2}\left[\alpha'^{2}_{ji}\,(J_{1i} - J_{3i}) + \alpha'^{1}_{ji} K_{2i} - \alpha'^{3}_{ji} K_{1i} + \Lambda'^{13}_{ji} - \Lambda'^{31}_{ji}\right]$$

$$a'^{22}_{ji} = \alpha'^{3}_{ji} K_{3i} - \alpha'^{1}_{ji} K_{1i} - \Lambda'^{33}_{ji} - \Lambda'^{11}_{ji}$$

$$a'^{23}_{ji} = \tfrac{1}{2}\left[\alpha'^{1}_{ji}\,(J_{3i} - J_{2i}) + \alpha'^{3}_{ji} K_{2i} - \alpha'^{2}_{ji} K_{3i} + \Lambda'^{23}_{ji} + \Lambda'^{32}_{ji}\right]$$

$$a'^{33}_{ji} = \alpha'^{1}_{ji} K_{1i} - \alpha'^{2}_{ji} K_{2i} - \Lambda'^{11}_{ji} - \Lambda'^{22}_{ji}$$

$$a'^{21}_{ji} = a'^{12}_{ji}\ ;\ a'^{31}_{ji} = a'^{13}_{ji}\ ;\ a'^{32}_{ji} = a'^{23}_{ji}$$

$$\boxed{KL,J}_{S} = \sum_{i=1}^{5}\Big[\alpha'^{1}_{ki}\,(\xi'^{23}_{lji} - \xi'^{32}_{lji}) + \alpha'^{1}_{li}\,(\xi'^{23}_{kji} - \xi'^{32}_{kji})$$
$$+ \alpha'^{2}_{ki}\,(\xi'^{31}_{lji} - \xi'^{13}_{lji}) + \alpha'^{2}_{li}\,(\xi'^{31}_{kji} - \xi'^{13}_{kji})$$
$$+ \alpha'^{3}_{ki}\,(\xi'^{12}_{lji} - \xi'^{21}_{lji}) + \alpha'^{3}_{li}\,(\xi'^{12}_{kji} - \xi'^{21}_{kji})\Big]$$
$$+ \sum_{i=1}^{5}\left[\alpha'^{1}_{ki}\ \alpha'^{2}_{ki}\ \alpha'^{3}_{ki}\right]\begin{bmatrix} a'^{11}_{ji} & a'^{12}_{ji} & a'^{13}_{ji} \\ a'^{21}_{ji} & a'^{22}_{ji} & a'^{23}_{ji} \\ a'^{31}_{ji} & a'^{32}_{ji} & a'^{33}_{ji} \end{bmatrix}\begin{Bmatrix} \alpha'^{1}_{li} \\ \alpha'^{2}_{li} \\ \alpha'^{3}_{li} \end{Bmatrix}$$
$$+ \sum_{i=1}^{5}\left[\alpha'^{1}_{ki}\ \alpha'^{2}_{ki}\ \alpha'^{3}_{ki}\right]\begin{bmatrix} (\Lambda'^{22}_{li}+\Lambda'^{33}_{li}) & -\Lambda'^{12}_{li} & -\Lambda'^{13}_{li} \\ -\Lambda'^{21}_{li} & (\Lambda'^{33}_{li}+\Lambda'^{11}_{li}) & -\Lambda'^{23}_{li} \\ -\Lambda'^{31}_{li} & -\Lambda'^{32}_{li} & (\Lambda'^{11}_{li}+\Lambda'^{22}_{li}) \end{bmatrix}\begin{Bmatrix} \alpha'^{1}_{ji} \\ \alpha'^{2}_{ji} \\ \alpha'^{3}_{ji} \end{Bmatrix}$$

(This equation continues on the next page)

$$+\sum_{i=1}^{3}\left[\alpha'^{1}_{ji}\ \alpha'^{2}_{ji}\ \alpha'^{3}_{ji}\right]\begin{bmatrix}(\Lambda'^{22}_{ki}+\Lambda'^{33}_{ki}) & -\Lambda'^{12}_{ki} & -\Lambda'^{13}_{ki}\\ -\Lambda'^{21}_{ki} & (\Lambda'^{33}_{ki}+\Lambda'^{11}_{ki}) & -\Lambda'^{23}_{ki}\\ -\Lambda'^{31}_{ki} & -\Lambda'^{32}_{ki} & (\Lambda'^{11}_{ki}+\Lambda'^{22}_{ki})\end{bmatrix}\begin{Bmatrix}\alpha'^{1}_{ji}\\ \alpha'^{2}_{ji}\\ \alpha'^{3}_{ji}\end{Bmatrix}$$

$$\boxed{KLJ} = \boxed{KLJ}\,F + \boxed{KLJ}\,S.$$

8 For aerodynamic parts of structural sections, using data submitted under this same heading,

$$\left.\begin{aligned} h^{1}_{ki} &= c^{1}_{k} + \gamma^{1}_{ki} + b^{2}_{k} z^{3}_{i} - b^{3}_{k} z^{2}_{i}\\ h^{2}_{ki} &= c^{2}_{k} + \gamma^{2}_{ki} + b^{3}_{k} z^{1}_{i} - b^{1}_{k} z^{3}_{i}\\ h^{3}_{ki} &= c^{3}_{k} + \gamma^{3}_{ki} + b^{1}_{k} z^{2}_{i} - b^{2}_{k} z^{1}_{i}\end{aligned}\right\}\ \text{for structural sections.}$$

$$\begin{aligned}\xi_{jih} = {} & n'^{1}_{ih}\Big(\sum_{r=1}^{3} e^{r}_{1i} h^{r}_{ji} + \sigma'^{1}_{jih} + \alpha'^{2}_{ji} v'^{3}_{ih} - \alpha'^{3}_{ji} v'^{2}_{ih}\Big)\\ & + n'^{2}_{ih}\Big(\sum_{r=1}^{3} e^{r}_{2i} h^{r}_{ji} + \sigma'^{2}_{jih} + \alpha'^{3}_{ji} v'^{1}_{ih} - \alpha'^{1}_{ji} v'^{3}_{ih}\Big)\\ & + n'^{3}_{ih}\Big(\sum_{r=1}^{3} e^{r}_{3i} h^{r}_{ji} + \sigma'^{3}_{jih} + \alpha'^{1}_{ji} v'^{2}_{ih} - \alpha'^{2}_{ji} v'^{1}_{ih}\Big)\end{aligned}$$

$$w_{ih} = [V_b^1\ V_b^2\ V_b^3] \begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix} \begin{Bmatrix} n_{ih}'^1 \\ n_{ih}'^2 \\ n_{ih}'^3 \end{Bmatrix}$$

$$V_b^r = V^r - V_a^r, \quad V^1 = V_c^1 - \Omega^2 z_c^3 + \Omega^3 z_c^2,$$

$$V^2 = V_c^2 - \Omega^3 z_c^1 + \Omega^1 z_c^3, \quad V^3 = V_c^3 - \Omega^1 z_c^2 + \Omega^2 z_c^1$$

In the three following summations, include only the terms for which $w_{ih} > 0$:

$$\left.\begin{array}{ll} 1. & R_{ji}^{tu} = \sum_{h=1}^{N_i} \xi_{jih}\, n_{ih}'^t\, n_{ih}'^u\, s_{ih} \\ 2. & S_{jki}^{tu} = \sum_{h=1}^{N_i} \xi_{jih}\, n_{ih}'^t\, T_{kih}'^u\, s_{ih} \\ 3. & T_{jki}^{t} = \sum_{h=1}^{N_i} \xi_{jih}\, \xi_{kih}\, n_{ih}'^t\, s_{ih} \end{array}\right\} \quad (t,u = 1,2,3)$$

$$\left[A_j^{rs}\right] = \sum_{i=1}^{5} \begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix} \begin{bmatrix} R_{ji}^{11} & R_{ji}^{12} & R_{ji}^{13} \\ R_{ji}^{21} & R_{ji}^{22} & R_{ji}^{23} \\ R_{ji}^{31} & R_{ji}^{32} & R_{ji}^{33} \end{bmatrix} \begin{bmatrix} e_{1i}^1 & e_{1i}^2 & e_{1i}^3 \\ e_{2i}^1 & e_{2i}^2 & e_{2i}^3 \\ e_{3i}^1 & e_{3i}^2 & e_{3i}^3 \end{bmatrix}$$

$$U^{t1}_{JKi} = S^{t1}_{JKi} + \alpha^{\prime 2}_{Ki} R^{t3}_{Ji} - \alpha^{\prime 3}_{Ki} R^{t2}_{Ji}$$

$$U^{t2}_{JKi} = S^{t2}_{JKi} + \alpha^{\prime 3}_{Ki} R^{t1}_{Ji} - \alpha^{\prime 1}_{Ki} R^{t3}_{Ji}$$

$$U^{t3}_{JKi} = S^{t3}_{JKi} + \alpha^{\prime 1}_{Ki} R^{t2}_{Ji} - \alpha^{\prime 2}_{Ki} R^{t1}_{Ji}$$

$$[B^{rs}_{JK}] = \sum_{i=1}^{s} \begin{bmatrix} e^1_{1i} & e^1_{2i} & e^1_{3i} \\ e^2_{1i} & e^2_{2i} & e^2_{3i} \\ e^3_{1i} & e^3_{2i} & e^3_{3i} \end{bmatrix} \begin{bmatrix} U^{11}_{JKi} & U^{12}_{JKi} & U^{13}_{JKi} \\ U^{21}_{JKi} & U^{22}_{JKi} & U^{23}_{JKi} \\ U^{31}_{JKi} & U^{32}_{JKi} & U^{33}_{JKi} \end{bmatrix} \begin{bmatrix} e^1_{1i} & e^2_{1i} & e^3_{1i} \\ e^1_{2i} & e^2_{2i} & e^3_{2i} \\ e^1_{3i} & e^2_{3i} & e^3_{3i} \end{bmatrix}$$

$$C^r_{JK} = \sum_{i=1}^{s} \sum_{t=1}^{3} e^r_{ti} T^t_{iK}$$

9. For the engines,

$$\Theta_J = \sum_{i=1}^{E} \left(h^1_{Ji} T_{xi} + h^2_{Ji} T_{yi} + h^3_{Ji} T_{zi} \right).$$

10 Equations of Motion.

$$\sum_{K=1}^{n} M_{JK} \ddot{q}^K = \sum_{r=1}^{3} \sum_{s=1}^{3} \Omega^r \Omega^s P_{rsJ} + \Theta_J$$

$$- \rho \sum_{r=1}^{3} \sum_{s=1}^{3} V^r_b V^s_b A^{rs}_J - (\omega_J)^2 M_{JJ} q^J$$

$$- 2\rho \sum_{K=1}^{n} \sum_{r=1}^{3} \sum_{s=1}^{3} V^r_b V^s_b B^{rs}_{JK} q^K - g_J \omega_J M_{JJ} \dot{q}^J$$

$$- 2\rho \sum_{K=1}^{n} \sum_{r=1}^{3} V^r_b C^r_{JK} q^K - \sum_{K=1}^{n} \sum_{L=1}^{n} \boxed{KL, J} \dot{q}^K q^L$$

Possibly, $q^j(t+\Delta t) = q^j(t) + \ddot{q}^j(t)\Delta t$

$$q^j(t+\Delta t) = q^j(t) + \dot{q}^j(t)\Delta t$$

If a better routine is available, use it.

ii For the computation of structural loads due to aerodynamic forces, including only the terms called for in data submittal 8 (p. 113) and for which $w_{ih} > 0$,

$$R_i'^{qtu} = \sum_h n_{ih}'^q n_{ih}'^t n_{ih}'^u S_{ih} \quad (q, t, u = 1, 2, 3)$$

$$U_{ki}'^{qt1} = \alpha_{ki}'^2 R_i'^{qt3} - \alpha_{ki}'^3 R_i'^{qt2} + \sum_h n_{ih}'^q n_{ih}'^t T_{kih}'^1 S_{ih}$$

$$U_{ki}'^{qt2} = \alpha_{ki}'^3 R_i'^{qt1} - \alpha_{ki}'^1 R_i'^{qt3} + \sum_h n_{ih}'^q n_{ih}'^t T_{kih}'^2 S_{ih}$$

$$U_{ki}'^{qt3} = \alpha_{ki}'^1 R_i'^{qt2} - \alpha_{ki}'^2 R_i'^{qt1} + \sum_h n_{ih}'^q n_{ih}'^t T_{ih}'^3 S_{ih}$$

$$T_{ki}'^{qt} = \sum_h n_{ih}'^q n_{ih}'^t \xi_{kih} S_{ih}$$

$$A_i'^{prs} = \sum_{q=1}^{3} \sum_{t=1}^{3} \sum_{u=1}^{3} e_{qi}^p e_{ti}^r e_{ui}^s R_i'^{qtu}$$

$$B_{ki}'^{prs} = \sum_{q=1}^{3} \sum_{t=1}^{3} \sum_{u=1}^{3} e_{qi}^p e_{ti}^r e_{ui}^s U_{ki}'^{qtu}$$

$$C_{ki}'^{pr} = \sum_{q=1}^{3} \sum_{t=1}^{3} e_{qi}^p e_{ti}^r T_k'^{qt}$$

$$\delta A'_{R_i} = -\rho \sum_{s=1}^{3}\sum_{t=1}^{3} V^s V^t A_i'^{rst} \quad (r=1,2,3)$$

$$\delta A^r_{E_i} = -2\rho \sum_{k=1}^{n}\sum_{s=1}^{3} V^s\left(\sum_{t=1}^{3} V^t B'^{rst}_{ki} q^k + C'^{rs}_{ki} q^k\right)$$

$$\delta A^r_{R\ell i} = -\rho \sum_{s=1}^{3}\sum_{t=1}^{3} V^s_\ell V^t_\ell A_i'^{rst}$$

$$\delta A^r_{E\ell i} = -2\rho \sum_{k=1}^{n}\sum_{s=1}^{3} V^s_\ell\left(\sum_{t=1}^{3} V^t_\ell B'^{rst}_{ki} q^k + C'^{rs}_{ki} q^k\right)$$

$$R_i''^{pqtu} = \sum_h v'^p_{ih} n'^q_{ih} n'^t_{ih} n'^u_{ih} S_{ih}$$

$$U_{ki}''^{pqt1} = \alpha'^2_{ki} R_i''^{pqt3} - \alpha'^3_{ki} R_i''^{pqt2} + \sum_h v'^p_{ih} n'^q_{ih} n'^t_{ih} T'^1_{kih} S_{ih}$$

$$U_{ki}''^{pqt2} = \alpha'^3_{ki} R_i''^{pqt1} - \alpha'^1_{ki} R_i''^{pqt3} + \sum_h v'^p_{ih} n'^q_{ih} n'^t_{ih} T'^2_{kih} S_{ih}$$

$$U_{ki}''^{pqt3} = \alpha'^1_{ki} R_i''^{pqt2} - \alpha'^2_{ki} R_i''^{pqt1} + \sum_h v'^p_{ih} n'^q_{ih} n'^t_{ih} T'^3_{kih} S_{ih}$$

$$T_{ki}''^{pqt} = \sum_h v'^p_{ih} n'^q_{ih} n'^t_{ih} \xi_{kih} S_{ih}$$

$$A_i''^{nrs} = \sum_{t=1}^{3}\sum_{u=1}^{3} e^r_{ti} e^s_{ui}\Big[e^n_{1i}\left(R_i''^{23tu} - R_i''^{32tu}\right) + e^n_{2i}\left(R_i''^{31tu} - R_i''^{13tu}\right) + e^n_{3i}\left(R_i''^{12tu} - R_i''^{21tu}\right)\Big]$$

$$B_{ki}''^{nrs} = \sum_{t=1}^{3}\sum_{u=1}^{3} e^r_{ti} e^s_{ui}\Big[e^n_{1i}\left(U_{ki}''^{23tu} - U_{ki}''^{32tu}\right) + e^n_{2i}\left(U_{ki}''^{31tu} - U_{ki}''^{13tu}\right) + e^n_{3i}\left(U_{ki}''^{12tu} - U_{ki}''^{21tu}\right)\Big]$$

$$C^{\prime\prime nr}_{Ki} = \sum_{t}^{3} e^{r}_{ti}\Big[e^{n}_{1i}\left(T^{\prime\prime 23t}_{Ki} - T^{\prime\prime 32t}_{Ki}\right)$$

$$+ e^{n}_{2i}\left(T^{\prime\prime 31t}_{Ki} - T^{\prime\prime 13t}_{Ki}\right) + e^{n}_{3i}\left(T^{\prime\prime 12t}_{Ki} - T^{\prime\prime 21t}_{Ki}\right)\Big]$$

$$N^{r}_{Ri} = -\rho \sum_{s=1}^{3}\sum_{t=1}^{3} V^{s} V^{t} A^{\prime\prime rst}_{i}$$

$$N^{r}_{Ei} = -2\rho \sum_{K=1}^{n}\sum_{s=1}^{3} V^{s}\left(\sum_{t=1}^{3} V^{t} B^{\prime\prime rst}_{Ki}\, q^{K} + C^{\prime\prime rs}_{Ki}\, \dot{q}^{K}\right)$$

$$N^{r}_{Rbi} = -\rho \sum_{s=1}^{3}\sum_{t=1}^{3} V^{s}_{b} V^{t}_{b} A^{\prime\prime rst}_{i}$$

$$N^{r}_{Ebi} = -2\rho \sum_{K=1}^{n}\sum_{s=1}^{3} V^{s}_{b}\left(\sum_{t=1}^{3} V^{t}_{b} B^{\prime\prime rst}_{Ki}\, q^{K} + C^{\prime\prime rs}_{Ki}\, \dot{q}^{K}\right)$$

$$\underline{SA}^{r}_{R} = \sum_{i} \underline{SA}^{r}_{Ri}$$

$$\underline{SA}^{r}_{E} = \sum_{i} \underline{SA}^{r}_{Ei}$$

$$\underline{SA}^{r}_{Rb} = \sum_{i} \underline{SA}^{r}_{Rbi}$$

$$\underline{SA}^{r}_{Eb} = \sum_{i} \underline{SA}^{r}_{Eb}.$$

$$\underline{MA}^{1}_{R\theta} = \sum_{i}\left(z^{2}_{i}\,\underline{SA}^{3}_{Ri} - z^{3}_{i}\,\underline{SA}^{2}_{Ri} + N^{1}_{Ri}\right)$$

$$\underline{MA}^{2}_{R\theta} = \sum_{i}\left(z^{3}_{i}\,\underline{SA}^{1}_{Ri} - z^{1}_{i}\,\underline{SA}^{3}_{Ri} + N^{2}_{Ri}\right)$$

$$\underline{MA}^{3}_{R\theta} = \sum_{i}\left(z^{1}_{i}\,\underline{SA}^{2}_{Ri} - z^{2}_{i}\,\underline{SA}^{1}_{Ri} + N^{3}_{Ri}\right)$$

$$\underline{MA}^{1}_{E\theta} = \sum_{i}\left(z^{2}_{i}\,\underline{SA}^{3}_{Ei} - z^{3}_{i}\,\underline{SA}^{2}_{Ei} + N^{1}_{Ei}\right)$$

$$\underline{MA}^{2}_{E\theta} = \sum_{i}\left(z^{3}_{i}\,\underline{SA}^{1}_{Ei} - z^{1}_{i}\,\underline{SA}^{3}_{Ei} + N^{2}_{Ei}\right)$$

$$MA^3_{E\theta} = \sum_i \left(z^1_i \, SA^2_{Ei} - z^2_i \, SA^1_{Ei} + N^3_{Ei} \right)$$

$$MA^1_{Rb\theta} = \sum_i \left(z^2_i \, SA^3_{Rbi} - z^3_i \, SA^2_{Rbi} + N^1_{Rbi} \right)$$

$$MA^2_{Rb\theta} = \sum_i \left(z^3_i \, SA^1_{Rbi} - z^1_i \, SA^3_{Rbi} + N^2_{Rbi} \right)$$

$$MA^3_{Rb\theta} = \sum_i \left(z^1_i \, SA^2_{Rbi} - z^2_i \, SA^1_{Rbi} + N^3_{Rbi} \right)$$

$$MA^1_{Eb\theta} = \sum_i \left(z^2_i \, SA^3_{Ebi} - z^3_i \, SA^2_{Ebi} + N^1_{Ebi} \right)$$

$$MA^2_{Eb\theta} = \sum_i \left(z^3_i \, SA^1_{Ebi} - z^1_i \, SA^3_{Ebi} + N^2_{Ebi} \right)$$

$$MA^3_{Eb\theta} = \sum_i \left(z^1_i \, SA^2_{Ebi} - z^2_i \, SA^1_{Ebi} + N^3_{Ebi} \right)$$

12. For the computation of structural loads due to thrust forces, including only the engines called for in data submittal 8 (p.113),

$$ST^1_R = \sum_i T_{xi}$$

$$ST^2_R = \sum_i T_{yi}$$

$$ST^3_R = \sum T_{zi}$$

$$MT^1_{R\theta} = \sum_i \left(z^2_i T_{zi} - z^3_i T_{yi} \right)$$

$$MT^2_{R\theta} = \sum_i \left(z^3_i T_{xi} - z^1_i T_{zi} \right)$$

$MJ^3_{RO} = \sum (\zeta^1_i T_{y_i} - \zeta^2_i T_{x_i}$

13 For the computation of structural loads due to inertial forces, including only the tanks, sections, and particles called for in data submittal 8 (p. 113).

$A^1 = \dot{V}^1_c + \Omega^2 V^3 - \Omega^3 V^2 - \dot{\Omega}^2 \zeta^3_c + \dot{\Omega}^3 \zeta^2_c$

$A^2 = \dot{V}^2_c + \Omega^3 V^1 - \Omega^1 V^3 - \dot{\Omega}^3 \zeta^1_c + \dot{\Omega}^1 \zeta^3_c$

$A^3 = \dot{V}^3_c + \Omega^1 V^2 - \Omega^2 V^1 - \dot{\Omega}^1 \zeta^2_c + \dot{\Omega}^2 \zeta^1_c$

$m'_i = \sum_h m_{ih}$

m_i if all particles of a section are used

$= M_{F_i}$ for tank i.

$\Upsilon'^r_i = \sum_h m_{ih} v'^r_{ih}$

$= 0$ if all particles are used, or for a tank

$\Psi'^r_{ki} = \sum_h m_{ih} \sigma'^r_{kih}$

$= 0$ if all particles are used or for a tank

$Y^r_i = m'_i \zeta^r_i + \sum_{s=1}^{3} e^r_{si} \Upsilon'^s_i$ for a structural section

$$Y^r \quad m_i \, \xi_F^r \quad \text{for a tank (that is, the fuel in the tank)}$$

$$Y^r \quad \sum_i Y_i^r$$

$$m' = \sum_i m'_i$$

$$Y_k^r = \sum_i \Big[m'_i \, n_{ki}^r + \sum_{s=1}^{3} e_{si}^r \Psi_{ki}'^s + e_{1i}^r (\alpha_{ki}'^2 \Upsilon_i'^3 - \alpha_{ki}'^3 \Upsilon_i'^2)$$

$$+ e_{2i}^r (\alpha_{ki}'^3 \Upsilon_i'^1 - \alpha_{ki}'^1 \Upsilon_i'^3) + e_{3i}^r (\alpha_{ki}'^1 \Upsilon_i'^2 - \alpha_{ki}'^2 \Upsilon_i'^1) \Big]$$

$$Y_{kl}^r = \sum_i \Big[e_{1i}^r (\alpha_{ki}'^2 \Psi_{li}'^3 - \alpha_{ki}'^3 \Psi_{li}'^2 + \alpha_{li}'^2 \Psi_{ki}'^3 - \alpha_{li}'^3 \Psi_{ki}'^2)$$

$$+ e_{2i}^r (\alpha_{ki}'^3 \Psi_{li}'^1 - \alpha_{ki}'^1 \Psi_{li}'^3 + \alpha_{li}'^3 \Psi \qquad {}_{li}'^1 \Psi_{ki}'^3)$$

$$+ e_{3i}^r (\alpha_{ki}'^1 \Psi_{li}'^2 - \alpha_{ki}'^2 \Psi_{li}'^1 + \alpha_{li}'^1 \Psi_{ki}'^2 - \alpha_{li}'^2 \Psi_{ki}'^1)$$

$$+ \frac{1}{2} \sum_{s=1}^{3} \sum_{t=1}^{3} e_{si}^r (\alpha_{ki}'^s \alpha_{li}'^t + \alpha_{li}'^s \alpha_{ki}'^t) \Upsilon_i'^t$$

$$- \Big(\sum_{s=1}^{3} e_{si}^r \Upsilon_i'^s\Big)\Big(\sum_{t=1}^{3} \alpha_{ki}'^t \alpha_{li}'^t\Big) \Big]$$

$$\underset{\sim}{SI}_R^1 = -\dot{A}^1 m' - \Omega^1 \sum_{s=1}^{3} \Omega^s Y^s + Y^1 \sum_{s=1}^{3} \Omega^s \Omega^s - \dot{\Omega}^2 Y^3 + \dot{\Omega}^3 Y^2$$

$$\underset{\sim}{SI}_R^2 = -A^2 m' - \Omega^2 \sum_{s=1}^{3} \Omega^s Y^s + Y^2 \sum_{s=1}^{3} \Omega^s \Omega^s - \dot{\Omega}^3 Y^1 + \dot{\Omega}^1 Y^3$$

$$\underset{\sim}{SI}_R^3 = -A^3 m' - \Omega^3 \sum_{s=1}^{3} \Omega^s Y^s + Y^3 \sum_{s=1}^{3} \Omega^s \Omega^s - \dot{\Omega}^1 Y^2 + \dot{\Omega}^2 Y^1$$

$$\underset{\sim}{SI}_E^1 = -2 \sum_{k=1}^{n} (\Omega^2 Y_k^3 - \Omega^3 Y_k^2) \dot{q}^k - \sum_{k=1}^{n} \sum_{l=1}^{n} Y_{kl}^1 q^k \dot{q}^l - \sum_{k=1}^{n} Y_k^1 \ddot{q}^k$$

$$\underset{\sim}{SI}_E^2 = -2 \sum_{k=1}^{n} (\Omega^3 Y_k^1 - \Omega^1 Y_k^3) \dot{q}^k - \sum_{k=1}^{n} \sum_{l=1}^{n} Y_{kl}^2 q^k \dot{q}^l - \sum_{k=1}^{n} Y_k^2 \ddot{q}^k$$

$$\underset{\sim}{S}\underset{\sim}{I}^{3}_{E} = -2\sum_{K=1}^{n}(\Omega^1 Y^2_K - \Omega^2 Y^1_K)\dot{q}^K - \sum_{K=1}^{n}\sum_{L=1}^{n} Y^3_{KL}\dot{q}^K\dot{q}^L - \sum_{K=1}^{n} Y^3_K \ddot{q}^K$$

$$\Pi_{rsi} = \sum_h m_{ih}\, \upsilon'^r_{ih}\, \upsilon'^s_{ih}$$

$= \Gamma'_{rsi}$ if all particles of a section are used

$= \Gamma'_{Frsi}$ for tank i.

$$\mathcal{Y}^{rs}_i = z^r_i\, \gamma'^s_i + \sum_{t=1}^{3} e^r_{ti}\, \Pi'_{tsi}$$

$$\Phi_{rs} = \sum_i \left(y^r_i z^s_i + \sum_{t=1}^{3} \mathcal{Y}^{rt}_i e^s_{ti} \right)$$

$$\mathrm{II}_{rs} = \delta_{rs}\sum_{t=1}^{3}\Phi_{tt} - \Phi_{rs}, \text{ where } \delta_{rs} = 1 \text{ when } r = s$$

$$= 0 \text{ when } r \neq s$$

$$M\underset{\sim}{I}^{1}_{R\theta} = A^2 Y^3 - A^3 Y^2 - \sum_{s=1}^{3}(\Omega^2 \mathrm{II}_{3s} - \Omega^3 \mathrm{II}_{2s})\Omega^s - \sum_{s=1}^{3}\mathrm{II}_{1s}\dot{\Omega}^s$$

$$M\underset{\sim}{I}^{2}_{R\theta} = A^3 Y^1 - A^1 Y^3 - \sum_{s=1}^{3}(\Omega^3 \mathrm{II}_{1s} - \Omega^1 \mathrm{II}_{3s})\Omega^s - \sum_{s=1}^{3}\mathrm{II}_{2s}\dot{\Omega}^s$$

$$M\underset{\sim}{I}^{3}_{R\theta} = A^1 Y^2 - A^2 Y^1 - \sum_{s=1}^{3}(\Omega^1 \mathrm{II}_{2s} - \Omega^2 \mathrm{II}_{1s})\Omega^s - \sum_{s=1}^{3}\mathrm{II}_{3s}\dot{\Omega}^s$$

$$\mathbb{A}'^{st}_{Ki} = \sum_h m_{ih}\, \upsilon'^s_{ih}\, \sigma'^t_{Kih}$$

$= \Lambda'^{st}_{Ki}$ if all particles are used, or for a tank

$$q'^{rs}_{Ki} = \delta_{rs}\sum_{t=1}^{3}\mathbb{A}'^{tt}_{Ki} - \mathbb{A}'^{rs}_{Ki}$$

$$W^r_{KL} = \sum_i \Big[\sum_{s=1}^{3}\sum_{t=1}^{3}\sum_{u=1}^{3} e^r_{si}\, e^t_{ui}\, z^t_i (\alpha'^s_{Ki}\, \psi'^u_{Li} - \psi'^s_{Li}\, \alpha'^u_{Ki})$$

(continued on next page)

$$+\mathbb{W}^{r}_{KL} + \sum_{s=1}^{3} \sum_{t=1}^{3} \alpha'^{s}_{Ki}\, \eta'^{\cdot t}_{\cdot\cdot}\, e^{r}_{ti}$$

$$-\frac{1}{2} e^{r}_{1i} \sum_{s=1}^{3} \left(\alpha'^{2}_{Li}\, \Pi'_{s3i} - \alpha'^{3}_{Li}\, \Pi'_{s2i}\right) \alpha'^{s}_{Ki}$$

$$-\frac{1}{2} e^{r}_{2i} \sum_{s=1}^{3} \left(\alpha'^{3}_{Li}\, \Pi'_{s1i} - \alpha'^{1}_{Li}\, \Pi'_{s3i}\right) \alpha'^{s}_{Ki}$$

$$-\frac{1}{2} e^{r}_{3i} \sum_{s=1}^{3} \left(\alpha'^{1}_{Li}\, \Pi'_{s2i} - \alpha'^{2}_{Li}\, \Pi'_{s1i}\right) \alpha'^{s}_{Ki} \Big],$$

where $\mathbb{W}^{1}_{KL} = \frac{1}{2}\sum_{s=1}^{3}\sum_{t=1}^{3}\left(\mathfrak{z}^{2}_{i} e^{3}_{si} - \mathfrak{z}^{3}_{i} e^{2}_{si}\right)\left(\alpha'^{s}_{Li} T'^{t}_{i} - T'^{s}_{i}\alpha'^{t}_{Li}\right)\alpha'^{t}_{Ki}$,

$$\mathbb{W}^{2}_{KL} = \frac{1}{2}\sum_{s=1}^{3}\sum_{t=1}^{3}\left(\mathfrak{z}^{3}_{i} e^{1}_{si} - \mathfrak{z}^{1}_{i} e^{3}_{si}\right)\left(\alpha'^{s}_{Li} T'^{t}_{i} - T'^{s}_{i}\alpha'^{t}_{Li}\right)\alpha'^{t}_{Ki},$$

and $\mathbb{W}^{3}_{KL} = \frac{1}{2}\sum_{s=1}^{3}\sum_{t=1}^{3}\left(\mathfrak{z}^{1}_{i} e^{2}_{si} - \mathfrak{z}^{2}_{i} e^{1}_{si}\right)\left(\alpha'^{s}_{Li} T'^{t}_{i} - T'^{s}_{i}\alpha'^{t}_{Li}\right)\alpha'^{t}_{Ki}$

$$\mathbb{Y}^{rs}_{Ki} = \mathfrak{z}^{r}_{i}\, \psi'^{s}_{Ki} + \sum_{t=1}^{3} e^{r}_{ti}\, \mathbb{A}'^{ts}_{Ki}$$

$$\mathbb{D}_{rsK} = \sum_{i}\Big[Y^{r}_{i} h^{s}_{Ki} + \sum_{t=1}^{3} \mathbb{Y}^{rt}_{Ki} e^{s}_{ti} + \left(\mathbb{Y}^{r2}_{i} e^{s}_{3i} - \mathbb{Y}^{r3}_{i} e^{s}_{2i}\right)\alpha'^{1}_{Ki}$$
$$+ \left(\mathbb{Y}^{r3}_{i} e^{s}_{1i} - \mathbb{Y}^{r1}_{i} e^{s}_{3i}\right)\alpha'^{2}_{Ki} + \left(\mathbb{Y}^{r1}_{i} e^{s}_{2i} - \mathbb{Y}^{r2}_{i} e^{s}_{1i}\right)\alpha'^{3}_{Ki}\Big]$$

$$\mathbb{P}_{rsK} = \delta_{rs} \sum_{t=1}^{3} \mathbb{D}_{ttK} - \mathbb{D}_{rsK}$$

$$H'_{rsi} = \delta_{rs} \sum_{t=1}^{3} \Pi'_{tti} - \Pi'_{rsi}$$

$$\mathbb{R}^{1}_{K} = \sum_{i}\Big[Y^{2}_{i} h^{3}_{Ki} - Y^{3}_{i} h^{2}_{Ki} + \sum_{r=1}^{3}\left(\mathbb{Y}^{2r}_{Ki} e^{3}_{ri} - \mathbb{Y}^{3r}_{Ki} e^{2}_{ri}\right)\Big]$$

$$\mathbb{R}^{2}_{K} = \sum_{i}\Big[Y^{3}_{i} h^{1}_{Ki} - Y^{1}_{i} h^{3}_{Ki} + \sum_{r=1}^{3}\left(\mathbb{Y}^{3r}_{Ki} e^{1}_{ri} - \mathbb{Y}^{1r}_{Ki} e^{3}_{ri}\right)\Big]$$

$$\mathbb{R}^{3}_{K} = \sum_{i}\Big[Y^{1}_{i} h^{2}_{Ki} - Y^{2}_{i} h^{1}_{Ki} + \sum_{r=1}^{3}\left(\mathbb{Y}^{1r}_{Ki} e^{2}_{ri} - \mathbb{Y}^{2r}_{Ki} e^{1}_{ri}\right)\Big]$$

$$R^{r}_{K} = \mathbb{R}^{r}_{K} + \sum_{i}\Big[\sum_{s=1}^{3}\sum_{t=1}^{3} e^{r}_{si} H'_{sti}\alpha'^{t}_{Ki} + \sum_{s=1}^{3}\sum_{t=1}^{3}\sum_{u=1}^{3} e^{r}_{si} e^{t}_{ui}\, \mathfrak{z}^{t}_{i}\left(\alpha'^{s}_{Ki} T'^{u}_{i} - T'^{s}_{i}\alpha'^{u}_{Ki}\right)\Big]$$

$$M\underset{\smile}{I}^{r}_{E\theta} = -2\left(\sum_{k=1}^{n}\sum_{s=1}^{3} \Omega^{s}\Gamma_{srk} q^{k} + \sum_{k=1}^{n}\sum_{l=1}^{n} W^{r}_{kl} q^{k} q^{l}\right) - \sum_{k=1}^{n} R^{r}_{k}\ddot{q}^{k}$$

For the computation of structural loads due to gravity,

$$S\underset{\smile}{G}^{r} = g^{r}_{a} m' \quad (r = 1,2,3)$$

$$M\underset{\smile}{G}^{1}_{\theta} = Y^{2} g^{3}_{a} - Y^{3} g^{2}_{a}$$

$$M\underset{\smile}{G}^{2}_{\theta} = Y^{3} g^{1}_{a} - Y^{1} g^{3}_{a}$$

$$M\underset{\smile}{G}^{3}_{\theta} = Y^{1} g^{2}_{a} - Y^{2} g^{1}_{a}$$

14. For the computation of Structural Loads to be printed,

$$y^{r}_{eg} = \jmath^{r}_{e} + \sum_{s=1}^{3} e^{r}_{se} v'^{s}_{eg}$$

Case 1

$$S^{r}_{R1} = S\underset{\smile}{A}^{r}_{R} + S\underset{\smile}{I}^{r}_{R} + S\underset{\smile}{G}^{r}$$

$$M^{1}_{R1} = M\underset{\smile}{A}^{1}_{R\theta} + M\underset{\smile}{I}^{1}_{R\theta} + M\underset{\smile}{G}^{1}_{\theta} - y^{2}_{eg}S^{3}_{R1} + y^{3}_{eg}S^{2}_{R1}$$

$$M^{2}_{R1} = M\underset{\smile}{A}^{2}_{R\theta} + M\underset{\smile}{I}^{2}_{R\theta} + M\underset{\smile}{G}^{2}_{\theta} - y^{3}_{eg}S^{1}_{R1} + y^{1}_{eg}S^{3}_{R1}$$

$$M^{3}_{R1} = M\underset{\smile}{A}^{3}_{R\theta} + M\underset{\smile}{I}^{3}_{R\theta} + M\underset{\smile}{G}^{3}_{\theta} - y^{1}_{eg}S^{2}_{R1} + y^{2}_{eg}S^{1}_{R1}$$

Case 2

$$S^r_{R2} = S^r_{R1} + ST^r_R$$

$$M^1_{R2} = M^1_{R1} + MT^1_{R\theta} - \psi^2_{eg} ST^3_R + \psi^3_{eg} ST^2_R$$

$$M^2_{R2} = M^2_{R1} + MT^2_{R\theta} - \psi^3_{eg} ST^1_R + \psi^1_{eg} ST^3_R$$

$$M^3_{R2} = M^3_{R1} + MT^3_{R\theta} - \psi^1_{eg} ST^2_R + \psi^2_{eg} ST^1_R$$

Case 3

$$S^r_{R\ell} = SA^r_{R\ell} + ST^r_R + SI^r_R + SG^r$$

$$M^1_{R\ell} = MA^1_{R\ell\theta} + MT^1_{R\theta} + MI^1_{R\theta} + MG^1_\theta - \psi^2_{eg} S^3_{R\ell} + \psi^3_{eg} S^2_{R\ell}$$

$$M^2_{R\ell} = MA^2_{R\ell\theta} + MT^2_{R\theta} + MI^2_{R\theta} + MG^2_\theta - \psi^3_{eg} S^1_{R\ell} + \psi^1_{eg} S^3_{R\ell}$$

$$M^3_{R\ell} = MA^3_{R\ell\theta} + MT^3_{R\theta} + MI^3_{R\theta} + MG^3_\theta - \psi^1_{eg} S^2_{R\ell} + \psi^2_{eg} S^1_{R\ell}$$

Case 4

$$S^r_E = SA^r_E + SI^r_E \quad \text{(do not print)}$$

$$S^r = S^r_{R2} + S^r_E$$

$$M^1 = M^1_{R2} + MA^1_{E\theta} + MI^1_{E\theta} - \psi^2_{eg} S^3_E + \psi^3_{eg} S^2_E$$

$$M^2 \quad M^2_{R2} + MA^2_{E\theta} + MI^2_{E\theta} - \psi^3_{eg} S^1_E + \psi^1_{eg} S^3_E$$

$$M^3 = M^3_{R2} + MA^3_{E\theta} + MI^3_{E\theta} - \psi^1_{eg} S^2_E + \psi^2_{eg} S^1_E$$

Case 5

$$S^r_{Eb} = SA^r_{Eb} + SI^r_E \quad (\text{do not print})$$

$$S^r_b = S^r_{Rb} + S^r_{Eb}$$

$$M^1_b = M^1_{Rb} + MA^1_{Eb\theta} + MI^1_{E\theta} - y^2_{eg} S^3_{Eb} + y^3_{eg} S^2_{Eb}$$

$$M^2_b = M^2_{Rb} + MA^2_{Eb\theta} + MI^2_{E\theta} - y^3_{eg} S^1_{Eb} + y^1_{eg} S^3_{Eb}$$

$$M^3_b = M^3_{Rb} + MA^3_{Eb\theta} + MI^3_{E\theta} - y^1_{eg} S^2_{Eb} + y^2_{eg} S^1_{Eb}$$

15. For the computation of accelerations and deflections, including only the points designated in data submittal 9 (p. 113),

$$y^r_{ih} = z^r_i + \sum_{s=1}^{3} e^r_{s1} \, u'^s_{ih}$$

Cases 1, 2, 3 (Use A^1, A^2, A^3 from 13, p. 147)

$$A^1_{Rih} = A^1 - y^1_{ih}(\Omega^2\Omega^2 + \Omega^3\Omega^3) + y^2_{ih}(\Omega^1\Omega^2 - \dot{\Omega}^3) + y^3_{ih}(\Omega^3\Omega^1 + \dot{\Omega}^2)$$

$$A^2_{Rih} = A^2 - y^2_{ih}(\Omega^3\Omega^3 + \Omega^1\Omega^1) + y^3_{ih}(\Omega^2\Omega^3 - \dot{\Omega}^1) + y^1_{ih}(\Omega^1\Omega^2 + \dot{\Omega}^3)$$

$$A^3_{Rih} = A^3 - y^3_{ih}(\Omega^1\Omega^1 + \Omega^2\Omega^2) + y^1_{ih}(\Omega^3\Omega^1 - \dot{\Omega}^2) + y^2_{ih}(\Omega^2\Omega^3 + \dot{\Omega}^1)$$

(Print the above 3 quantities for each point)

Cases 4&5

$$\Gamma^r_{kih} = h^r_{ki} + \sum_{s=1}^{3} e^r_{si}\,\sigma'^s_{ki} \quad \begin{vmatrix} e^r_{1i} & \alpha'^1_{ki} & \upsilon'^1_{i} \\ e^r_{2i} & \alpha'^2_{ki} & \upsilon'^2_{i} \\ e^r_{3i} & \alpha'^3_{ki} & \upsilon'^3_{i} \end{vmatrix}$$

$$D^r_{ih} = \sum_{k=1}^{n} \Gamma^r_{kih}\, q^k \quad \text{(print)}$$

$$\upsilon^r_{Eih} = \sum_{k=1}^{n} \Gamma^r_{kih}\, \dot{q}^k$$

$$w^r_{Eih} = \sum_{k=1}^{n} \Gamma^r_{kih}\, \ddot{q}^k$$

$$\Phi^r_{kLih} = \frac{1}{2}\sum_{s=1}^{3}\sum_{t=1}^{3} e^r_{si}\,\alpha'^t_{ki}\left(\alpha'^s_{Li}\,\upsilon'^t_{ih} - \alpha'^t_{Li}\,\upsilon'^s_{ih}\right)$$

$$+ \begin{vmatrix} e^r_{1i} & \alpha'^1_{ki} & \sigma'^1_{Lih} \\ e^r_{2i} & \alpha'^2_{ki} & \sigma'^2_{Lih} \\ e^r_{3i} & \alpha'^3_{ki} & \sigma'^3_{Lih} \end{vmatrix}$$

$$\chi^r_{Eih} = 2\sum_{k=1}^{n}\sum_{L=1}^{n} \Phi^r_{kLih}\, \dot{q}^k \dot{q}^L$$

$$A^1_{Eih} = 2\left(\Omega^2 \upsilon^3_{Eih} - \Omega^3 \upsilon^2_{Eih}\right) + w^1_{Eih} + \chi^1$$

$$A^2_{Eih} = 2\left(\Omega^3 \upsilon^1_{Eih} - \Omega^1 \upsilon^3_{Eih}\right) + w^2_{Eih} + \chi^2$$

$$A^3_{Eih} = 2\left(\Omega^1 \upsilon^2_{Eih} - \Omega^2 \upsilon^1_{Eih}\right) + w^3_{Eih} + \chi$$

$$a^r_{ih} = A^r_{Rih} - A^r_{Eih} \quad \text{(print)}$$